Force distribution in a randomly perturbed lattice of identical particles with $1/r^2$ pair interaction

Andrea Gabrielli
Istituto dei Sistemi Complessi–CNR, Via dei Taurini 19, 00185 Rome, Italy and SMC-INFM, Physics Department, University "La Sapienza" of Rome, 00185 Rome, Italy

Thierry Baertschiger
Dipartimento di Fisica, Università "La Sapienza," 00185 Rome, Italy and Istituto dei Sistemi Complessi–CNR, Via dei Taurini 19, 00185 Rome, Italy

Michael Joyce
Laboratoire de Phyisique Nucléaire et Hautes Energies, Université Pierre et Marie Curie–Paris 6, UMR 7585, F-75005 Paris, France

Bruno Marcos
Dipartimento di Fisica, Università "La Sapienza," 00185 Rome, Italy, and Istituto dei Sistemi Complessi–CNR, Via dei Taurini 19, 00185 Rome, Italy

Francesco Sylos Labini
E. Fermi Center, Via Panisperna 89 A, Compendio del Viminale, 00184 Rome, Italy and Istituto dei Sistemi Complessi–CNR, Via dei Taurini 19, 00185 Rome, Italy
(Received 6 March 2006; revised manuscript received 22 June 2006; published 9 August 2006)

We study the statistics of the force felt by a particle in the class of a spatially correlated distribution of identical pointlike particles, interacting via a $1/r^2$ pair force (i.e., gravitational or Coulomb), and obtained by randomly perturbing an infinite perfect lattice. We specify the conditions under which the force on a particle is a well-defined stochastic quantity. We then study the small displacements approximation, giving both the limitations of its validity and, when it is valid, an expression for the force variance. The method introduced by Chandrasekhar to find the force probability density function for the homogeneous Poisson particle distribution is extended to shuffled lattices of particles. In this way, we can derive an approximate expression for the probability distribution of the force over the full range of perturbations of the lattice, i.e., from very small (compared to the lattice spacing) to very large where the Poisson limit is recovered. We show in particular the qualitative change in the large-force tail of the force distribution between these two limits. Excellent accuracy of our analytic results is found on detailed comparison with results from numerical simulations. These results provide basic statistical information about the fluctuations of the interactions (i) of the masses in self-gravitating systems like those encountered in the context of cosmological N-body simulations, and (ii) of the charges in the ordered phase of the one-component plasma, the so-called Coulomb or Wigner crystal.

DOI: 10.1103/PhysRevE.74.021110 PACS number(s): 02.50.−r, 05.40.−a, 61.43.−j

I. INTRODUCTION

The study of the statistical properties of the force felt by a particle in a gas, and exerted by all the other particles of the system through pair interactions, can provide useful insights into the thermodynamics or dynamics of physical systems in many different contexts and applications. Some important examples are given by (i) the distribution of the gravitational force in a spatial distribution of pointlike masses in cosmological and astrophysical applications [1–3], (ii) the statistics of molecular and dipolar interactions [4] in a gas of particles, (iii) the theory of interacting defects in condensed-matter physics [5], and (iv) the contact force distribution in granular materials [6,7].

The seminal work in this field is due to Chandrasekhar [1] and deals mainly with the probability density function (PDF) of the gravitational force in a homogeneous spatial Poisson distribution of identical pointlike masses. Specifically, the PDF of the force is found to be given exactly by the Holtzmark distribution, which is a three-dimensional (3D) fat tailed Lévy distribution [8]. In [2,4,9,10], approximated generalizations, in different physical contexts, of this approach can be found for more complex pointlike particle systems obtained by perturbing a homogeneous Poisson particle distribution.

Here we present a study of the probability distribution of the total gravitational (or Coulomb) force for a specific class of spatial particle distributions (i.e., point processes): three-dimensional *shuffled lattices*, i.e., lattices in which each particle is randomly displaced from its lattice position with a PDF of the displacement $p(\mathbf{u})$. This study can be very useful for applications in both solid-state physics (e.g., in the case of Coulomb or dipolar pair interaction) [4] and in astrophysics and cosmology. In particular, in solid-state physics it is known that under suitable physical conditions a gas of identical pointlike charges (immersed in a uniform background with opposite sign of charge, conserving the global charge neutrality), i.e., the so-called one-component plasma (OCP) [11], crystallizes giving rise to a state of matter called Coulomb or Wigner crystal [12]. At finite temperature, such a perfect crystal is perturbed by the presence of phonons (i.e., lattice vibrations) that create nonzero Coulomb interactions

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among particles. As we show in this paper, there is a strong link between the statistical properties of the total force acting on a generic particle and those of the lattice deformations. Moreover, the shape of the force probability distribution is related to the "effective" locality of the interactions, i.e., to the relative importance of the neighbors of a given particle in determining the total force.

In the cosmological context, large N-body numerical simulations of the dynamics of self-gravitating identical massive particles constitute an essential instrument in the study of the problem of structure formation in the Universe, starting from small initial mass fluctuations. These simulations are usually performed by starting from initial conditions, which are suitably perturbed simple lattices [13,14]. Knowledge of the statistical properties of the gravitational force felt by a single particle in such a distribution is important in several respects. First and foremost, it allows one to quantify the relative importance of the contribution to the force due to the immediate neighborhood of each particle compared with that due to far away particles. This is essential both in assessing the importance of effects coming from the finite size of simulations, and also in understanding the extent to which a mean-field-type description of the dynamics, in which the effects of strong local fluctuations due to discreteness are neglected, is valid. Detailed knowledge of the force distribution at early times also gives useful information for the numerical discretization of the dynamics, as the force acting on particles is crucial in determining the temporal resolution. Analytical results on these quantities are also potentially very useful as instruments for controlling the precision of numerical calculations of the force in different approximation schemes. In this paper we consider, as already mentioned, the case in which the perturbations to the particles initially on the lattice are uncorrelated, while in the cosmological context they are generically correlated. The results we use here will be directly applied in a parallel study, which we will report in a forthcoming paper [15], of the gravitational dynamics from these shuffled lattice initial conditions.

We will discuss briefly in our conclusions the kinds of questions that may be addressed in this context, and for the case of the slightly more complex initial conditions used in cosmology, using the formal results and methods developed in this paper.

Quite generally, Chandrasekhar [1] showed that the gravitational force acting on each particle can be seen as due to the superposition of two different contributions: the former (a sort of *fluid* term) is due to the system as a whole, and the latter is due to the influence of the immediate neighborhood of the particle and therefore depends on how a spatial mass distribution (e.g., a fluid) is represented through a pointlike particle system. The former is a smoothly varying function of position while the latter is subject to relatively rapid fluctuations. We will show that this is true also for the case of a shuffled lattice, even though some important differences with respect to the case of the homogeneous Poisson case are present when the shuffling is very small due the extreme uniformity of the particle distribution even at small scales. When, instead, the shuffling is sufficiently large, we recover approximately the same behavior as in the Poisson case.

The rest of the paper is organized as follows. In Sec. II, the principal one- and two-point correlation properties of point processes and in particular of a shuffled lattice are briefly reviewed. In Sec. III, we give the statistical definition of the global gravitational force acting on each particle specifying the conditions under which this is a well-defined stochastic quantity. In particular, we discuss the problem posed by the infinite volume limit and the necessity of introducing a compensating uniform negative background mass density for the statistical definiteness of the gravitational force in this limit. Here we point out also that this study and the definitions given are valid in both the cases of gravitational and Coulomb pair interaction. In Sec. IV, we study the stochastic total gravitational force acting on a particle to linear order in the random displacements. In this way, we identify two different contributions to this force: the former comes from the displacements of the pointlike sources keeping the particle on which we calculate the force at its original lattice position, and the latter comes from the displacement of this particle with respect to the rest of the lattice. The first contribution is dominated by the particles in the neighborhood of the particle on which we calculate the force, while the second can be seen as a force due to the system as a continuous mass distribution. In Sec. V, the variance of the gravitational force is calculated in the above approximation and its meaning is discussed in relation to the form $p(\mathbf{u})$ of the displacement PDF. In particular, we find the fundamental difference of the behavior between the two cases in which the particle displacements are limited or not to the elementary lattice cell. In Sec. VI, in order to go beyond a study of the statistical properties of the gravitational force limited to the first and second moment, we briefly report some previously known results about the PDF of the force in two different situations: the first is the exact solution of Chandrasekhar for the case of a three-dimensional homogeneous Poisson particle distribution, and the second concerns the PDF of the total force in one-dimensional (1D) shuffled lattices of particles interacting via generic power-law pair interactions. In Sec. VII, we use the results exposed in the previous section to develop and discuss an approximate evaluation in the manner of Chandrasekhar of the PDF (i.e., of all the moments) of the total force acting on a given particle. This provides much useful information about the exact PDF of the force. The approximation becomes better and better when the typical permitted displacements of the particles approach and go beyond the limit of the elementary lattice cell. In Sec. VIII, we perform a comparison of the theoretical and analytical results found in the preceding sections with the results obtained by numerical simulations. The agreement is shown to be very good in general, despite the fact that the three-dimensional problem of the gravitational force in a shuffled lattice is not exactly solvable. In Sec. IX, we summarize the main results of this work and draw some concluding remarks on their utility.

II. STATISTICAL PROPERTIES OF A SHUFFLED LATTICE

Let us introduce some basic notation that will be useful in the rest of the paper. For the correlation functions and spectra, we will use the notation adopted in cosmology as our study will find there its main field of application. Given a spatial distribution of pointlike particles in a cubic volume V (we indicate with V both the space region in which the system is defined and its volume; in this paper we are interested in the limit $V\to\infty$) with equal mass m (i.e., a so-called *point process* [10,18]), the microscopic mass density function is

$$\rho(\mathbf{x}) = m\sum_i \delta(\mathbf{x}-\mathbf{x}_i), \tag{1}$$

where $\delta(\mathbf{x})$ is the 3D Dirac delta function, $\mathbf{x}_i$ is the position of the ith particle of the system, and the sum is over all the particles of the system. Clearly the microscopic number density is simply given by $n(\mathbf{x})=\rho(\mathbf{x})/m$. Let us suppose that $n_0>0$ is the average number density. Consequently, the average mass density is simply $\rho_0=n_0 m$. The power spectrum (PS)[1] of such a system is defined as

$$\mathcal{P}(\mathbf{k}) \equiv \lim_{V\to\infty}\frac{\langle|\delta\hat{n}(\mathbf{k};V)|^2\rangle}{n_0^2 V} \equiv \lim_{V\to\infty}\frac{\langle|\delta\hat{\rho}(\mathbf{k};V)|^2\rangle}{\rho_0^2 V}, \tag{2}$$

where $\langle\cdots\rangle$ indicates the average over all the realizations of the point process, and

$$\delta\hat{n}(\mathbf{k};V) = \int_V d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}[n(\mathbf{x})-n_0]$$

is the Fourier integral[2] in the volume V of the number density contrast $[n(\mathbf{x})-n_0]$. Note that $\mathcal{P}(\mathbf{k})$ does not depend on m. Therefore, without loss of generality we fix $m=1$ for the sake of simplicity and use $n(\mathbf{x})$ for both the number and the mass density function.

The connected two-point correlation function [10,19], also called the covariance function, is defined in general as

$$\tilde{\xi}(\mathbf{x},\mathbf{x}') = \frac{\langle n(\mathbf{x})n(\mathbf{x}')\rangle}{n_0^2} - 1.$$

In the case of statistical translational invariance (or spatial homogeneity), it is a function only of the vector distance $(\mathbf{x}-\mathbf{x}')$, i.e., $\tilde{\xi}(\mathbf{x},\mathbf{x}')=\tilde{\xi}(\mathbf{x}-\mathbf{x}')$ and satisfies the relation

$$\tilde{\xi}(\mathbf{x}) = \mathcal{F}^{-1}[\mathcal{P}(\mathbf{k})]$$

and conversely

$$\mathcal{P}(\mathbf{k}) = \mathcal{F}[\tilde{\xi}(\mathbf{x})]. \tag{3}$$

In general, the correlation function $\tilde{\xi}(\mathbf{x})$ has the following structure:

$$\tilde{\xi}(\mathbf{x}) = \frac{\delta(\mathbf{x})}{n_0} + \xi(\mathbf{x}), \tag{4}$$

where $\delta(\mathbf{x})/n_0$ is the singular "diagonal part" of the covariance function due to the fact that the microscopic density has an infinite discontinuity on any particle, and $\xi(\mathbf{x})$ [often denoted by $h(\mathbf{x})$ in the statistical physics literature] is the smooth "off-diagonal" part giving the real two-point correlation between the positions of different particles. The function $\langle n(\mathbf{x})\rangle_p=n_0[1+\xi(\mathbf{x})]$ gives the average conditional density of particles seen from a spatial point occupied by a particle [10] (where $\langle\cdots\rangle_p$ indicates a conditional ensemble average). This implies that $\xi(\mathbf{x})\geq -1$ for all $\mathbf{x}$ as the average density of particles seen by any particle of the system cannot be negative. Moreover, at $\mathbf{x}$ such that $\xi(\mathbf{x})>0$ or <0, the conditional average density is, respectively, larger or smaller than n_0, and this gives the physical meaning of *positive* and *negative* density-density correlations for particle distributions at a given distance.

The off-diagonal part $\xi(\mathbf{x})$ vanishes identically only for the class of homogeneous Poisson point processes in which each pointlike particle occupies a randomly chosen spatial point with no correlation with the other particles of the system [10,18]. For this reason, this is typically considered as the paradigmatic example of a spatially uniform and homogeneous particle distribution. Note that for this class of point processes, one has from Eq. (3) $\mathcal{P}(\mathbf{k})=1/n_0$ at all $\mathbf{k}$. Another important quantity to characterize mass (i.e., number) fluctuations in particle distributions is the *number variance* in spheres of radius R,

$$\sigma_N^2(R) = \langle N^2(R)\rangle - \langle N(R)\rangle^2,$$

where $N(R)$ is the number of particles in a sphere $S(R)$ of radius R. Clearly $\langle N(R)\rangle=n_0\|S(R)\|$, where $\|S(R)\|=(4\pi/3)R^3$ is the volume of the sphere. It is possible to show [10] that in general $\sigma_N^2(R)$ can be related to the PS in the following way:

$$\frac{\sigma_N^2(R)}{\langle N(R)\rangle^2} = \int\frac{d^3k}{(2\pi)^3}\mathcal{P}(\mathbf{k})\hat{W}^2(k;R)$$

with $\hat{W}(k,R)=\mathcal{F}[W(\mathbf{x},R)]$ and where $W(\mathbf{x},R)$ is the window function of the sphere, equal to unity if $\mathbf{x}$ is inside the sphere centered at the origin and zero otherwise. In particular, one is interested in the large-R scaling behavior of $\sigma_N^2(R)$. For instance, in the Poisson case $\sigma_N^2(R)\sim\|S(R)\|\sim R^3$ (this is the famous law of Poisson noise). In general, if $\mathcal{P}(k)\sim k^n$ at small k (for the Poisson case obviously $n=0$) one has $\sigma_N^2(R)\sim R^m$ at large R with $m=3-n$ for $-3<n\leq 1$ (with logarithmic corrections for $n=1$), and $m=2$ for $n>1$. Therefore, for $n>0$, integrated mass fluctuations scale slower than in the Poisson case. For this reason, this class of mass distributions is called *superhomogeneous* [10,19].

Let us now come to the class of system we study in this paper. In general, a perturbed lattice can be built by applying a stochastic displacement to each particle initially belonging to a regular lattice (e.g., simple cubic). If $\mathbf{R}$ is the generic lattice site, the density function $n(\mathbf{x})$ for a regular lattice reads

[1]In condensed-matter physics it is usually called *structure factor*.

[2]In the limit $V\to\infty$, we adopt in general the following usual normalizations for the Fourier transform: $\mathcal{F}[g(\mathbf{x})]=\int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}g(\mathbf{x})$ is the Fourier transform of the function $g(\mathbf{x})$, and $\mathcal{F}^{-1}[\hat{g}(\mathbf{k})]=\frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\hat{g}(\mathbf{k})$ is the inverse Fourier transform.

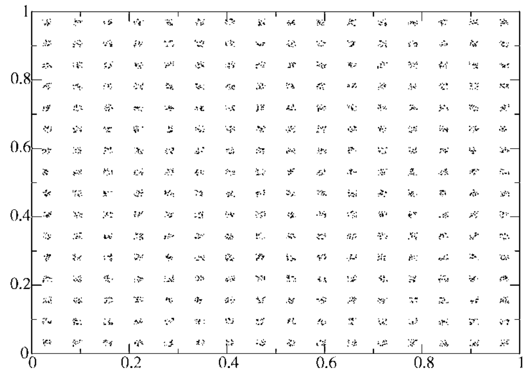

FIG. 1. The figure provides the projection on the x-y plane of a 3D *shuffled* simple cubic lattice of 16^3 particles in a cubic unitary volume. In this case $p(\mathbf{u})=\Pi_{i=1}^{3}f(u_i)$, where $f(u_i)=\theta(\Delta-|u_i|)/(2\Delta)$ (as in the simulations considered in Sec. VIII) with $2\Delta=\ell/4$ (i.e., each particle is displaced well inside its elementary lattice cell).

$$n(\mathbf{x}) = \sum_{\mathbf{R}} \delta(\mathbf{x}-\mathbf{R}), \tag{5}$$

where the sum is over all the lattice sites. If, moreover, $\mathbf{u}(\mathbf{R})$ is the displacement applied to the particle initially at $\mathbf{R}$, the function $n(\mathbf{x})$ for such a SL will be

$$n(\mathbf{x}) = \sum_{\mathbf{R}} \delta(\mathbf{x}-\mathbf{R}-\mathbf{u}(\mathbf{R})). \tag{6}$$

In Fig. 1, we present a typical configuration of a 3D perturbed cubic lattice projected on one of the lattice planes. The perturbed lattice is said to be *uncorrelated* if the displacements applied to different particles are taken independent of each other. We call such a system a *shuffled* lattice (SL). This means that $p^{(2)}(\mathbf{u}(\mathbf{R}),\mathbf{u}(\mathbf{R}'))=p[\mathbf{u}(\mathbf{R})]p[\mathbf{u}(\mathbf{R}')]$, where $p^{(2)}(\mathbf{u}(\mathbf{R}),\mathbf{u}(\mathbf{R}'))$ is the *joint* PDF of the displacements applied to two different particles respectively initially at $\mathbf{R}$ and $\mathbf{R}'$, respectively, and $p(\mathbf{u})$ is the PDF of a single displacement. Without loss of generality, in the following we limit the discussion to the symmetric case in which $p(\mathbf{u})=p(-\mathbf{u})$. Clearly the average over all the possible realizations of the displacement field coincides with the average over all the possible realizations of the point process. Therefore, we will use the notation

$$\langle g(\mathbf{u}_1,\ldots,\mathbf{u}_l)\rangle = \int d^3u_1\cdots d^3u_l g(\mathbf{u}_1,\ldots,\mathbf{u}_l)p(\mathbf{u}_1)\cdots p(\mathbf{u}_l)$$

for the average of any function of the l displacements applied, respectively, to l different points.

By calling $\hat{p}(\mathbf{k})$ the *characteristic function* of a single stochastic displacement, i.e., its Fourier transform (FT), we can write the PS of the final point process as [20]

$$\mathcal{P}(\mathbf{k}) = \frac{1}{n_0}[1-|\hat{p}(\mathbf{k})|^2] + (2\pi)^3|\hat{p}(\mathbf{k})|^2 \sum_{\mathbf{H}\neq\mathbf{0}} \delta(\mathbf{k}-\mathbf{H}), \tag{7}$$

where $\mathbf{H}$ is the generic vector of the reciprocal lattice [21] of the real-space lattice giving the initial particle configuration, and

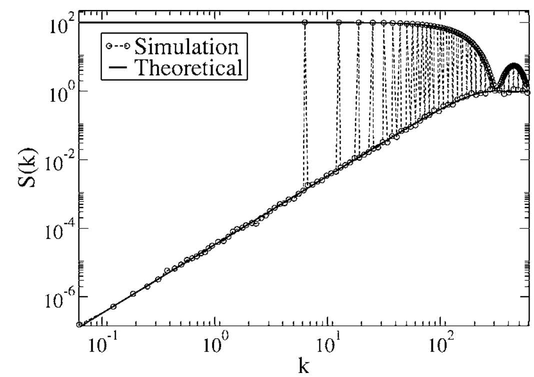

FIG. 2. Comparison between the PS $S(k)=n_0^2\mathcal{P}(k)$ measured in numerical simulations (circles) of a 1D SL (average over 10^3 realizations of the same SL) with the theoretical prediction (continuous curve) given by Eq. (7) in the case in which the one displacement PDF $p(u)=\theta(\Delta-|u|)/(2\Delta)$ with $\delta=\Delta/\ell=1/50$. In order to represent appropriately the modulated Dirac delta functions contribution to $S(k)$, we have normalized their amplitude to a value 10^2 for the unperturbed lattice and compared their modulation with the theoretical modulation $|\hat{p}(k)|^2$ multiplied also by 10^2 (top continuous curve). For values of k different from Bragg peaks, numerical results are compared with the continuous part of the theoretical PS $[1-|\hat{p}(\mathbf{k})|^2]/n_0$.

$$\mathcal{P}_{\rm in}(\mathbf{k}) = (2\pi)^3 \sum_{\mathbf{H}\neq\mathbf{0}} \delta(\mathbf{k}-\mathbf{H})$$

is the PS of this initial regular configuration. A 1D example of such a PS is given in Fig. 2, in which the exact formula (7) is compared with the PS evaluated in numerical simulations of the SL, showing a perfect agreement. Note that if the applied displacement field is isotropic, $p(\mathbf{u})$ depends only on $u=|\mathbf{u}|$ and $\hat{p}(\mathbf{k})$ only on $k=|\mathbf{k}|$. However, as the initial particle configuration is a regular lattice, for a SL there is not full translational invariance. Therefore, rigorously speaking, $\mathcal{F}^{-1}[\mathcal{P}(\mathbf{k})]$ cannot be seen as the the two-point correlation function of the particle system, but only as the average of $\tilde{\xi}(\mathbf{x}_0+\mathbf{x},\mathbf{x}_0)$ over $\mathbf{x}_0$ as random in the elementary lattice cell.[3]

It is evident from Eq. (7) that the random shuffling $\{\mathbf{u}(\mathbf{R})\}$ in general does not erase completely in the PS $\mathcal{P}(\mathbf{k})$ the presence of the so-called *Bragg peaks* (i.e., the sum of δ functions) for each reciprocal-lattice vector $\mathbf{H}\neq\mathbf{0}$, of $\mathcal{P}_{\rm in}(\mathbf{k})$, but only modulates their amplitude proportionally to $|\hat{p}(\mathbf{H})|^2$, and adds a continuous contribution typical of stochastic particle distributions $[1-|\hat{p}(\mathbf{k})|^2]/n_0$. Around $k=0$ (more precisely in the whole *first Brillouin zone* [21]), $\mathcal{P}_{\rm in}(\mathbf{k})=\mathbf{0}$ identically [i.e., we can say that around $k=0$, $\mathcal{P}_{\rm in}(\mathbf{k})\sim k^\infty$] Thus, from this point of view, regular lattices can be seen as the class of the most superhomogeneous particle distributions. As is clear from Eq. (7), in this region $\mathcal{P}(\mathbf{k})$ is determined by only the behavior of the displacement characteristic function $\hat{p}(\mathbf{k})$. In particular, even though the lattice is strictly aniso-

[3]Note that this means that when there is not statistical translational invariance, $\mathcal{P}(\mathbf{k})$ contains less information than the two-point correlation function $\tilde{\xi}(\mathbf{x},\mathbf{x}')$.

tropic, this implies that if the displacement field is statistically isotropic, the final SL has isotropic mass fluctuations at large scales (i.e., for $k \to 0$). By assuming $p(\mathbf{u}) = p(u)$, it is now simple to show [20] that for any PDF at small k we have

$$\hat{p}(k) = 1 - Ak^{\alpha} + o(k^{\alpha}), \quad (8)$$

where $\alpha=2$ and $A=\overline{u^2}/6$ if $\overline{u^2}$ is finite [where $\overline{f(\mathbf{u})} = \int d^3u p(\mathbf{u}) f(\mathbf{u})$], while $\alpha=\beta-3$ and $A>0$ if $p(u)$ decays as $u^{-\beta}$ at large u with $\beta<5$ [note that $\beta>3$ for any PDF $p(u)$ in three dimensions], and therefore $\overline{u^2}$ is infinite.

In this paper, we focus our attention on the class of SL where, as written above, the applied displacements are statistically uncorrelated.

III. DEFINITION OF THE GRAVITATIONAL FORCE ON A PARTICLE IN A PERTURBED LATTICE

Let us now consider basic questions concerning the definition of the gravitational force acting on a particle in an infinite perturbed lattice. As aforementioned, we assume that (i) all particles have mass $m=1$, (ii) the average number density is $n_0 = \ell^{-3}$, where ℓ is the lattice spacing, (iii) the microscopic number density is given by Eq. (6), and (iv) we choose the units so that the gravitational constant $G=1$. Let us suppose for simplicity that the initial position of the particle on which we evaluate the gravitational field is the origin of coordinates (i.e., $\mathbf{R}=\mathbf{0}$). The total gravitational force acting on it is

$$\mathbf{F} = \sum_{\mathbf{R}\neq\mathbf{0}} \frac{\mathbf{R} + \mathbf{u}(\mathbf{R}) - \mathbf{u}(0)}{|\mathbf{R} + \mathbf{u}(\mathbf{R}) - \mathbf{u}(0)|^3}, \quad (9)$$

where the sum is over all the particles initially at the lattice sites $\mathbf{R}\neq\mathbf{0}$. The same sum can be simply rewritten as

$$\mathbf{F} = \int_V d^3x n'(\mathbf{x}) \frac{\mathbf{x} - \mathbf{u}(0)}{|\mathbf{x} - \mathbf{u}(0)|^3}, \quad (10)$$

where V is the system volume, and

$$n'(\mathbf{x}) = n(\mathbf{x}) - \delta(\mathbf{x} - \mathbf{u}(0)), \quad (11)$$

with $n(\mathbf{x})$ given by Eq. (6).

Note that by simply inverting the sign of the pair interaction, and therefore of the total force from attractive to repulsive, and substituting the identical masses with identical charges, Eq. (9) gives the total repulsive Coulomb force $\mathbf{F}_C$ in a perturbed Coulomb lattice [12] on the point-charge at $\mathbf{u}(0)$ and generated by all the other point charges at $\mathbf{R} + \mathbf{u}(\mathbf{R})$,

$$\mathbf{F}_C = \sum_{\mathbf{R}\neq\mathbf{0}} \frac{\mathbf{u}(0) - \mathbf{R} - \mathbf{u}(\mathbf{R})}{|\mathbf{R} + \mathbf{u}(\mathbf{R}) - \mathbf{u}(0)|^3} = \int_V d^3x n'(\mathbf{x}) \frac{\mathbf{u}(0) - \mathbf{x}}{|\mathbf{x} - \mathbf{u}(0)|^3}. \quad (12)$$

Consequently, all the results in this paper can be directly applied to the statistics of the repulsive force in a shuffled Coulomb lattice.

We are interested in the limit $V\to\infty$ of Eq. (10), where we mean by this limit that the volume V tends to the whole of $\mathbb{R}^3$. It is well known in different physical contexts [21,22] that, if $n_0>0$, the infinite volume limit of lattice sums such as those in Eq. (9) or Eq. (12) is in fact not well defined because these sums are only conditionally convergent in the limit $V\to\infty$, i.e., their result depends on the order in which the single terms are summed. In many physical applications, however, as in the case of the Coulomb lattice [12], this sum is regularized automatically by the presence in the physical system of a uniform background charge density with opposite sign with respect to that of the point charges and such that there is overall charge neutrality. Once the attractive force $\mathbf{F}_b$ of the background on the point charges is taken into account, by adding the corresponding term

$$\mathbf{F}_b(\mathbf{u}_0) = n_0 \int_V d^3x \frac{\mathbf{x} - \mathbf{u}_0}{|\mathbf{x} - \mathbf{u}_0|^3}$$

to Eq. (12), then the total Coulomb force acting on a given point charge is finite and its value is independent of the way in which the infinite volume limit is taken.[4] To clarify this point, let us consider the following system: a density of point charges $n(\mathbf{x})$ given by Eq. (1) with $m=1$ embedded in a uniform background charge density $-n_0 = -\langle n(\mathbf{x})\rangle$. Therefore, the local charge density at the point $\mathbf{x}$ will be $\delta n(\mathbf{x}) = n(\mathbf{x}) - n_0$. The force acting on a probe charge at the point $\mathbf{y}$ of the space and generated by the total charge in the volume V around it will be

$$\mathbf{F}(\mathbf{y};V) = \int_V d^3x \delta n(\mathbf{x}) \frac{\mathbf{y} - \mathbf{x}}{|\mathbf{y} - \mathbf{x}|^3}. \quad (13)$$

Let us now assume that the PS of $n(\mathbf{x})$, and therefore of $\delta n(\mathbf{x})$, has the behavior $\mathcal{P}(\mathbf{k}) \sim k^a$ at small k, with $a>-3$ (in three dimensions). Under these hypotheses (see, e.g., [20]) it is simple to show that

$$\left\langle \left| \int_V d^3x \delta n(\mathbf{x}) \right|^2 \right\rangle \sim V^{b/3}$$

with $b=3-a$ if $a<1$ and $b=2$ if $a\geq 1$. From Eq. (13), we then expect in general that $\mathbf{F}(\mathbf{y}) = \mathbf{F}(\mathbf{y}; V\to+\infty)$ is a well defined stochastic quantity (i.e., a stochastic quantity whose statistics is well defined and not depending on how the compact volume V is sent to infinity) if $a>-1$.[5] Indeed for these values of a, fluctuations in the density contrast $\delta n(\mathbf{x})$ generate in the infinite volume limit quadratic fluctuations in the force $\mathbf{F}$ that are size-independent. This is, in particular, the case for the SL we consider in this paper, for which we have shown that $a>0$ for any displacement PDF $p(\mathbf{u})$ [see Eq.

[4]This is true provided V is a compact set of $\mathbb{R}^3$, containing both the point charges and the uniform opposite charged background.

[5]Note, however, that the *difference* of the force between any two points of the space is a well defined stochastic quantity for any statistically homogeneous particle distribution with $a>-3$, i.e., when the mass density PS is well defined for $k\to 0$ (i.e., integrable). In systems with $-3<a\leq -1$, the force acting on the center of mass of any subregion would be divergent in the infinite volume limit, but this is not relevant for the evolution of the relative distance of the particles.

(8)]. Given that the limit $V \to \infty$ of Eq. (13) does not depend on the way in which the limit is performed, we can choose for simplicity to take the volume V symmetric with respect to the point $\mathbf{y}$ where the force is computed. In such a volume, the contribution to the force $\mathbf{F}(\mathbf{y};V)$ from the background vanishes by symmetry. Consequently, with this choice of the volume V, the force $\mathbf{F}(\mathbf{y})$ coincides with the limit of the sum (13) with $\delta n(\mathbf{x})$ replaced by $n(\mathbf{x})$ (i.e., summing only over the particles).

This, in particular, implies that for the SL particle distribution we will study here, in the presence of an oppositely charged neutralizing background, we can evaluate the well defined global force $\mathbf{F}$ acting on the point charge in $\mathbf{u}(0)$ simply using Eq. (12) where the sum is over all the charges contained in a sphere $S[r;\mathbf{u}(0)]$ of radius r centered on $\mathbf{u}(0)$ and then taking the limit $r \to \infty$. Note that, on the other hand, the limit $r \to \infty$ of Eq. (12) using instead the sphere $S[r;0]$ centered on the point 0, where the considered particle was *before* the displacement, does not give the full force $\mathbf{F}$ on the point $\mathbf{u}(0)$. To obtain it the background contribution, which now is different from zero, must be added.

In the rest of this paper, for convenience, we will choose to take the infinite volume limit in the former way, i.e., symmetrically in spheres about the considered particle, also in the treatment of the gravitational problem, i.e., for Eq. (9). It is possible in fact to show that the presence of an analogous background with a negative mass density (now exerting a repulsive force on the massive particles) comes out naturally when the motion of a particle is described in comoving coordinates starting from the exact equations of general relativity in a quasi-uniform expanding Universe (see, e.g., [23]). In pure Newtonian gravity, instead, such a background does not exist and has to be introduced artificially to regularize the problem (*Jean's swindle*, see [22]). The results given here for the statistics of the force $\mathbf{F}$ have thus to be understood as strictly valid in the presence of such a compensating background.

In [24], a simple estimation was given of the contribution of the first six nearest neighbors (NN) of the particle to Eq. (9) for a simple cubic lattice. We show now that instead the sum of Eq. (9) on a sphere of radius r around the central particle can be seen as the sum of two different contributions: the first "asymmetric" one is due to the self-shuffling $\mathbf{u}(0)$ of the center particle from the initial $\mathbf{R}=\mathbf{0}$, and can be seen as induced by the system as a whole. The second "symmetric" term is due to the shuffling of all the other particles, and is dominated by the contribution of the first six NN.

IV. THE SMALL DISPLACEMENTS BEHAVIOR OF THE FORCE

In this section, we will give the approximate expression of $\mathbf{F}$ obtained through a linearization in the particle displacements. The statistical meaning and the limitations on taking averages of powers of this linearized expression will be discussed in the next section. To simplify our computation, but without loss of generality of the results, we limit the discussion to those lattices with a cubic symmetry: i.e., simple cubic, body centered cubic, and face centered cubic lattices.

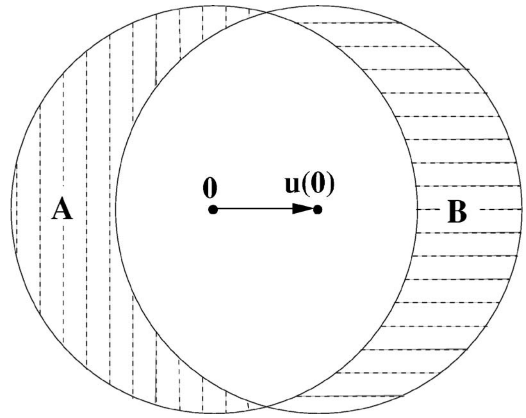

FIG. 3. The two overlapping spheres $S(r;0)$ and $S[r;\mathbf{u}(0)]$ of radius r, centered, respectively, at 0 and $\mathbf{u}(0)$, are represented. The integral in Eq. (16) over $\delta S[r;\mathbf{u}(0)]$ corresponds to the integral over the region B in the figure (indicated with horizontal dashed lines) minus the integral over the region A (indicated with vertical dashed lines). This clarifies the dipolar nature of this term.

As shown above, let us rewrite Eq. (9) as

$$\mathbf{F} = \lim_{r\to\infty} \int_{S[r;\mathbf{u}(0)]} n'(\mathbf{x}) \frac{\mathbf{x}-\mathbf{u}(0)}{|\mathbf{x}-\mathbf{u}(0)|^3} d^3x, \qquad (14)$$

where the integral is over the sphere $S[r;\mathbf{u}(0)]$ defined above, and $n'(\mathbf{x})$ is given by Eq. (11). In this section, we are interested in the linear contribution in the displacement field $\mathbf{F}^{(l)}$ to Eq. (14) for small displacements. Note that

$$d\mathbf{F}(\mathbf{x}) = n'(\mathbf{x}) \frac{\mathbf{x}-\mathbf{u}(0)}{|\mathbf{x}-\mathbf{u}(0)|^3} d^3x$$

is the force contribution coming from the volume element d^3x around $\mathbf{x}$. Therefore, we can rewrite Eq. (14) as

$$\mathbf{F} = \lim_{r\to\infty} \mathbf{F}[r;\mathbf{u}(0)] = \lim_{r\to\infty} \int_{S[r;\mathbf{u}(0)]} d\mathbf{F}(\mathbf{x}). \qquad (15)$$

We now write

$$\int_{S[r;\mathbf{u}(0)]} d\mathbf{F}(\mathbf{x}) = \int_{S(r;0)} d\mathbf{F}(\mathbf{x}) + \int_{\delta S[r;\mathbf{u}(0)]} d\mathbf{F}(\mathbf{x}), \quad (16)$$

where $S(r;0)$ is the sphere of radius r centered on the lattice point $\mathbf{R}=\mathbf{0}$, and the integration over $\delta S[r;\mathbf{u}(0)]$ means the integration over the portion of $S[r;\mathbf{u}(0)]$ not included in $S(r;0)$ minus the one over the portion of $S(r;0)$ not included in $S[r;\mathbf{u}(0)]$[6] (see Fig. 3).

Note that these portions of sphere both have the same volume, which is proportional to $|\mathbf{u}(0)|$. Consequently, we expect that this second integral will give a force contribution of order $\mathbf{u}(0)$.

Let us start by evaluating the first term in Eq. (16),

[6]This coincides with the integration over the sphere $S[r;\mathbf{u}(0)]$ minus the integration on the sphere $S(r;0)$.

$$\mathbf{F}_S(r) \equiv \int_{S(r;0)} d\mathbf{F}(\mathbf{x}),$$

to first order in the displacements. It can be written as Eq. (9), where the sum is over all the particles contained in the sphere $S(r;0)$ with the exception of the one at $\mathbf{u}(0)$. We thus denote this kind of sum by $\Sigma_{\mathbf{R}\neq 0}^{R\leq r}$, i.e.,

$$\mathbf{F}_S(r) = \sum_{\mathbf{R}\neq 0}^{R\leq r} \frac{\mathbf{R}+\mathbf{u}(\mathbf{R})-\mathbf{u}(0)}{|\mathbf{R}+\mathbf{u}(\mathbf{R})-\mathbf{u}(0)|^3}. \tag{17}$$

We are interested in the contribution $\mathbf{F}_S^{(l)}(r)$ to this quantity at linear order in the displacement field. Performing a Taylor expansion of Eq. (17) to first order in the (relative) displacements, we can write

$$\mathbf{F}_S^{(l)}(r) = \sum_{\mathbf{R}\neq 0}^{R\leq r} \mathbf{f}_{\mathbf{R}}, \tag{18}$$

with

$$\mathbf{f}_{\mathbf{R}} = \frac{3[\mathbf{u}(0)\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}-\mathbf{u}(0)}{R^3} + \frac{\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}}{R^3}, \tag{19}$$

where $\hat{\mathbf{R}}=\mathbf{R}/R$ is the unit vector in the direction of the lattice site $\mathbf{R}$. The quantity $\mathbf{F}_S^{(l)}(r)$ will be a good approximation to $\mathbf{F}_S(r)$ if all the (relative) displacements are sufficiently small (see the next section). Equations (18) and (19) show that, at first order, the contribution to the force separates into a part due to the displacement of the particle on which we calculate the force, and a part due to the displacement of the particle originally at $\mathbf{R}$. Let us write

$$\mathbf{f}_{\mathbf{R}} = \mathbf{f}_{\mathbf{R}}^o + \mathbf{f}_{\mathbf{R}}^s \tag{20}$$

with

$$\mathbf{f}_{\mathbf{R}}^o = \frac{-\mathbf{u}(0)+3[\mathbf{u}(0)\cdot\hat{\mathbf{R}}]}{R^3}, \tag{21}$$

$$\mathbf{f}_{\mathbf{R}}^s = \frac{\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}}{R^3}. \tag{22}$$

It is simple to show, for the class of lattices we consider, that

$$\sum_{\mathbf{R}\neq\mathbf{0}}^{R\leq r} \mathbf{f}_{\mathbf{R}}^o \equiv \mathbf{0} \tag{23}$$

when the sum is taken up to include all the lattice sites $\mathbf{R}\neq\mathbf{0}$ distributed symmetrically with respect to $\mathbf{R}=\mathbf{0}$ up to a given distance. The key points to show this are (i) to rewrite the sum in Eq. (23) as a sum over all the permitted values of $R=|\mathbf{R}|\leq r$ of the sums over all the sites with fixed R; (ii) to consider that for the sub-sum over all the sites $\mathbf{R}$, with fixed $R=|\mathbf{R}|$, we can write

$$\sum_{\mathbf{R}}^{|\mathbf{R}|=R} \mathbf{f}_{\mathbf{R}}^o = \frac{1}{R^3}\sum_{\mathbf{R}}^{|\mathbf{R}|=R}\{-\mathbf{u}(0)+3[\mathbf{u}(0)\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}\};$$

and (iii) to note that

$$3\sum_{\mathbf{R}}^{|\mathbf{R}|=R}[\mathbf{u}(0)\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}} = \sum_{\mathbf{R}}^{|\mathbf{R}|=R}(\hat{\mathbf{R}}\cdot\hat{\mathbf{R}})\mathbf{u}(0) = \sum_{\mathbf{R}}^{|\mathbf{R}|=R}\mathbf{u}(0)$$

from which Eq. (23) follows. Therefore, we can conclude that

$$\mathbf{F}_S^{(l)}(r) = \sum_{\mathbf{R}\neq 0}^{R\leq r}\frac{\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}}{R^3} \tag{24}$$

and we will call simply

$$\mathbf{F}_S^{(l)} = \lim_{r\to\infty}\mathbf{F}_S^{(l)}(r).$$

Let us consider now the second term in Eq. (16), which we call $\mathbf{F}_A(r)$,

$$\mathbf{F}_A(r) = \int_{\delta S[r;\mathbf{u}(0)]} d^3x \frac{n'(\mathbf{x})[\mathbf{x}-\mathbf{u}(0)]}{|\mathbf{x}-\mathbf{u}(0)|^3}. \tag{25}$$

Since we wish to evaluate the above integral to linear order in $\mathbf{u}(0)$, and the volume of integration is already of order $|\mathbf{u}(0)|$, we can substitute $n'(\mathbf{x})$ with its average [the point $\mathbf{u}(0)$ is not included in the volume of integration] $n_0=\ell^{-3}$ and put $\mathbf{u}(0)=0$ in the integrand. It is then straightforward to show that, taking the limit $r\to\infty$ and working to first order in $\mathbf{u}(0)$, we have

$$\mathbf{F}_A \simeq \mathbf{F}_A^{(l)} = \frac{4\pi}{3}n_0\mathbf{u}(0), \tag{26}$$

where we have simply called $\mathbf{F}_A=\lim_{r\to\infty}\mathbf{F}_A(r)$ and $\mathbf{F}_A^{(l)}$ its approximation at linear order in the displacement. Note that, as discussed in the previous section, this quantity can be seen as the force exerted by the negative background contained in the sphere $S(r;0)$ on the particle at $\mathbf{u}(0)$.

Let us summarize before proceeding further in the next section. We have now seen that, to first order in the displacements, we can write

$$\mathbf{F}^{(l)} = \mathbf{F}_A^{(l)} + \mathbf{F}_S^{(l)} = \frac{4\pi}{3}n_0\mathbf{u}(0) + \lim_{r\to\infty}\sum_{\mathbf{R}\neq 0}^{R\leq r}\frac{\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}}{R^3}, \tag{27}$$

where the sum is taken in the sphere $S(r;0)$. As explained in Sec. III, Eq. (27) gives the well defined infinite volume limit of the force, to first order in the displacements, generated by the system composed both by the particles and the uniform negative background density. Note that Eq. (27) can be interpreted as either the force on the particle at $\mathbf{u}(0)$ due only to the other particles contained in the sphere $S[r;\mathbf{u}(0)]$ (with $r\to\infty$), with no contribution from the background due to symmetry reasons, or the sum of the force on the particle at $\mathbf{u}(0)$ coming from both the other particles contained in the sphere $S(r;0)$ (with $r\to\infty$) and the background in the same sphere.

V. SMALL DISPLACEMENTS VARIANCE OF THE FORCE

We now turn to the problem of the variance of the force $\langle F^2\rangle$. Considering Eq. (27), one might think that, if the variance of the displacements $\overline{u^2} \ll \ell^2$, then the right-hand side of Eq. (27) can be used to evaluate the variance of the force. This is not, however, as we will see, always true. In fact, we will show below, if $p(\mathbf{u})>0$ for $u>\ell/2$ in such a way to permit at least a pair (and therefore an infinite number of pairs) of particles of the SL to be found arbitrarily close to one another, then $\langle F^2\rangle$ diverges due to the divergence of the pair interaction for vanishing separation. This clarifies the meaning and the validity of this "small displacement" approximation given by Eq. (27): in order to use it to calculate $\langle F^2\rangle$, it is not sufficient to have $\overline{u^2} \ll \ell^2$. It is instead necessary (and sufficient) that all the displacements are smaller than half the elementary lattice cell size ℓ [i.e., that the support of $p(\mathbf{u})$ is completely contained in the elementary lattice cell] so that each particle has a positive minimal separation from its first nearest neighbor.

In this section, we will suppose that this is the case, i.e., displacements are limited to a region contained in the elementary lattice cell. Given this hypothesis, because of the mutual statistical independence of the displacements applied to different particles, the average quadratic force acting on each particle, to order $\overline{u^2}$, is then simply

$$\begin{aligned}\langle F^2\rangle &\simeq \langle|\mathbf{F}^{(l)}|^2\rangle\\ &= \langle|\mathbf{F}_A^{(l)}|^2\rangle + \langle|\mathbf{F}_S^{(l)}|^2\rangle\\ &= \left(\frac{4\pi}{3}n_0\right)^2\overline{u^2} + \lim_{r\to\infty}\sum_{\mathbf{R}\neq 0}^{R\leq r}\frac{\langle|\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}|^2\rangle}{R^6}.\end{aligned} \tag{28}$$

If, moreover, we assume that not only the displacements of different particles but also the different components of the displacement of a single particle are uncorrelated, it is simple to show that, for the class of lattices considered, we have

$$\langle|\mathbf{u}(\mathbf{R})-3[\mathbf{u}(\mathbf{R})\cdot\hat{\mathbf{R}}]\hat{\mathbf{R}}|^2\rangle = 2\langle|\mathbf{u}(\mathbf{R})|^2\rangle = 2\overline{u^2}. \tag{29}$$

With these hypotheses, we can therefore write

$$\langle|\mathbf{F}^{(l)}|^2\rangle = \left[\left(\frac{4\pi}{3}n_0\right)^2 + 2\sum_{\mathbf{R}\neq\mathbf{0}}\frac{1}{R^6}\right]\overline{u^2}, \tag{30}$$

where the sum is now over all the lattice vectors $\mathbf{R}\neq\mathbf{0}$. Performing the lattice sum in Eq. (30), for a simple cubic lattice, we find

$$\begin{aligned}2\overline{u^2}\sum_{\mathbf{R}\neq 0}\frac{1}{R^6} &= 12\left(\frac{1}{\ell^3}\right)^2\overline{u^2}(c_{1\mathrm{NN}}+c_{2\mathrm{NN}}+c_{3\mathrm{NN}}+\cdots)\\ &\approx 16.1\left(\frac{1}{\ell^3}\right)^2\overline{u^2},\end{aligned} \tag{31}$$

where $c_{i\mathrm{NN}}$ is the relative contribution to the sum $\Sigma_{\mathbf{R}\neq 0}\frac{1}{R^6}$ of the set of ith nearest-neighbor lattice sites of the origin $\mathbf{R}=\mathbf{0}$, normalized such that $c_{1\mathrm{NN}}=1$ (giving $c_{2\mathrm{NN}}=1/4$ and $c_{3\mathrm{NN}}=4/81$). With these approximations, we then find for a simple cubic lattice

$$\sqrt{\langle F^2\rangle} \simeq \sqrt{\langle|\mathbf{F}^{(l)}|^2\rangle} = \sqrt{\langle|\mathbf{F}_S^{(l)}|^2\rangle + \langle|\mathbf{F}_A^{(l)}|^2\rangle} = \alpha n_0 U_0, \tag{32}$$

where $U_0=\sqrt{\overline{u^2}}$ and

$$\alpha \approx \sqrt{16.1+\left(\frac{4\pi}{3}\right)^2} \approx 5.86. \tag{33}$$

Hence we can draw a first conclusion: while in the case of a homogeneous Poisson particle distribution [1] the gravitational force acting on a given particle is dominated by the first nearest neighbor, in the present case it is dominated by two terms: the former is a global term $\mathbf{F}_A$ due to the displacement with respect to the rest of the system of the particle on which we calculate the force, and the latter, $\mathbf{F}_S$, is mainly due to the set of the first-nearest-neighbors lattice sites, which all lie at "almost" the same distance. As shown explicitly below, because of the symmetries of the initial lattice, the net gravitational force in a SL is clearly very depressed, for small displacements, with respect to the single-nearest-neighbor contribution of the Poisson case with the same average density n_0.

VI. USEFUL RESULTS ON THE PROBABILITY DISTRIBUTION OF THE FORCE

In the next section, we will generalize to our case the method introduced by Chandrasekhar [1] for calculating the PDF $P(\mathbf{F})$ of the gravitational force $\mathbf{F}$ in a 3D homogeneous Poisson particle distribution [10]. As a starting point, we briefly report in this scetion Chandrasekhar's results for this latter case. The specific form of $P(\mathbf{F})$ in this case is called the *Holtzmark distribution*, and for this reason we will denote it $P_H(\mathbf{F})$. Subsequently, we give a brief presentation of the exact derivation of the $P(\mathbf{F})$ in a 1D SL for a general power-law pair interaction. Finally, we proceed to generalize Chandrasekhar's method to the 3D SL by introducing some ad hoc approximations. These results, together with those presented in previous sections on the small displacement approximation, will provide us with a good qualitative comprehension of the behavior of $P(\mathbf{F})$ in a 3D SL when the shape of $p(\mathbf{u})$ is varied, and in particular when one passes from displacements limited to an elementary lattice cell to larger maximal displacements.

A. Gravitational force PDF in a homogeneous 3D Poisson particle distribution

In this subsection, we give a brief account of the force PDF $P_H(\mathbf{F})$ for a three-dimensional homogeneous Poisson particle distribution which, as aforementioned, is called the Holtzmark distribution. The exact mathematical derivation of this PDF can be found in [1]. One considers a homogeneous Poisson particle distribution in a cubic volume with average number density n_0. As written above, its two-point correlation function is simply

$$\tilde{\xi}(\mathbf{x}) = \frac{\delta(\mathbf{x})}{n_0},$$

i.e., there is no correlation between the position of different particles. Because of the total absence of correlations in the positions of different particles, the PDF of the force/field at a point is the same whether the point is occupied or not by a particle.

While it is not possible to write an explicit expression for $P_H(\mathbf{F})$, is is possible for its FT $A_H(\mathbf{q})=\mathcal{F}[P_H(\mathbf{F})]$, i.e., for the characteristic function of $\mathbf{F}$ [1],

$$A_H(\mathbf{q}) \equiv e^{-n_0 C_H(q)} = \exp\left[-n_0 \frac{4(2\pi q)^{3/2}}{15}\right]. \tag{34}$$

Chandrasekhar's derivation of this relation is based on the following observations: (i) The definition of $P_H(\mathbf{F})$ is

$$P_H(\mathbf{F}) = \int d^3x_1 \cdots d^3x_N p_c^{(N)}(\mathbf{x}_1, \ldots, \mathbf{x}_N)\,\delta\left(\mathbf{F} - \sum_{i=1}^{N} \frac{\mathbf{x}_i}{x_i^3}\right), \tag{35}$$

where $p_c^{(N)}(\mathbf{x}_1,\ldots,\mathbf{x}_N)$ is the joint conditional PDF of the positions of the whole set of N particles in the system volume V seen by the particle, at the origin of the chosen axes, experiencing the force; (ii) from the definition of a homogeneous Poisson point process, one has simply $p_c^{(N)} \times(\mathbf{x}_1,\ldots,\mathbf{x}_N)=(1/V)^N$ (i.e., it coincides with the joint PDF of the positions with no condition on the occupation of the origin). By using this in Eq. (35) and taking the FT together with the limit $(N,V)\to+\infty$ with fixed $n_0=N/V$, it is simple to recover Eq. (34).

Note that $A_H(\mathbf{q})$ is a typical example of a characteristic function of a *Levy stable distribution* [25] with exponent $3/2$. The fact that $A_H(\mathbf{q})$ depends only on $q=|\mathbf{q}|$ means that $P_H(\mathbf{F})$ depends only on $F=|\mathbf{F}|$ (the particle distribution is statistically isotropic). Since $A_H(\mathbf{q})$ is not analytic at $q=0$, $P_H(\mathbf{F})$ has a power-law tail at large F. Specifically, as $A_H(\mathbf{q})\simeq[1-4n_0(2\pi q)^{3/2}/15]$, we have $P_H(\mathbf{F})\sim F^{-9/2}$. This shows that moments $\langle F^\alpha\rangle$ of order $\alpha \geqslant 3/2$ diverge.[7]

In particular, the variance of the force diverges. It is simple to see that this is due to the small-scale divergence of the pair gravitational interaction together with the fact that particles can be found arbitrarily close to one another. We then expect the same large-F scaling behavior of $P(\mathbf{F})$ for a SL when the support of $p(\mathbf{u})$ is larger than the elementary lattice size, as, just as in the Poisson distribution, one can then find an infinite number of pairs of particles with members arbitrarily close to one another. We show this below both with the exact results in one dimension in Sec. VI B, and with the approximate approach in the manner of Chandrasekhar in Sec. VII.

We first recall the limiting behaviors of $P_H(\mathbf{F})$, which can be deduced directly from Eq. (34),

$$W_H(F) \sim \frac{4}{3\pi F_*^3} F^2 \quad \text{for } F \to 0^+, \tag{36a}$$

$$W_H(F) \sim \frac{15}{8}\sqrt{\frac{2}{\pi}} F_*^{3/2} F^{-5/2} \quad \text{for } F \to \infty \tag{36b}$$

with

$$F_* = 2\pi\left(\frac{4n_0}{15}\right)^{2/3},$$

and where $W_H(F)=4\pi F^2 P_H(\mathbf{F})$ is the PDF of the modulus of $\mathbf{F}$. The quantity F_* can be seen as the typical force acting on a particle and is called the *normal field* in [1]. It is also important to note that, in the Poisson case for large values of F, the $W_H(F)$ is well approximated by the PDF of the modulus of the force due only to the NN particle [10],

$$W_{\rm NN}(F) = 2\pi n_0 F^{-5/2} \exp\left(-\frac{4\pi n_0 F^{-3/2}}{3}\right). \tag{37}$$

This means that the main contribution to the force felt by a particle in a homogeneous Poisson distribution comes from the first NN particle, and is due to the small distance divergence of the pair gravitational interaction.

B. Exact results for the 1D SL

Before introducing an approximate approach *à la Chandrasekhar* for the SL in 3D, we give some exact results obtained for the force PDF in an analogous 1D SL of particles interacting via a power-law interaction as presented in [26].

Let us consider a 1D SL, i.e., a set of $2N+1$ pointlike particles of unitary mass placed at the points $x_m=m\ell+u_m$ with $m=-N,\ldots,N$ of the segment $[-L/2,L/2]$, where $\ell=L/(2N+1)$ is the lattice spacing (and $n_0=1/\ell$ is the average particle density) and u_m is the displacement of the mth particle from the lattice position $m\ell$. We assume that all the u_m are extracted from the same PDF $p(u)$ independently of one another, and that the particles interact via the attractive pair force,

$$f(x) = \frac{x}{|x|^{\alpha+1}},$$

where x is the particle separation and $\alpha>0$.

Therefore, the particle at $x_0=u_0$ feels the total force

$$F = \sum_{m\neq 0}^{-N,N} \frac{m\ell + u_m - u_0}{|m\ell + u_m - u_0|^{\alpha+1}}. \tag{38}$$

Note that F is not a sum of statistically independent terms as each of these terms depend on two displacements, one of which is always u_0. By considering that the PDF of x_m is simply given by $p(x_m-m\ell)$, we can formally write the PDF $P(F)$ of the stochastic force F as [26]

$$P(F) = \int \cdot\cdot \int_{-\infty}^{+\infty} \left[\prod_{m=-\infty}^{+\infty} dx_m p(x_m - m\ell)\right] \times \delta\left(F - \sum_{m\neq 0}^{-\infty,+\infty} \frac{x_m - x_0}{|x_m - x_0|^{\alpha+1}}\right), \tag{39}$$

in which we have taken directly the (symmetric) limit $N\to\infty$ with ℓ fixed. By taking the FT in F of Eq. (39), one can write the characteristic function of the stochastic force as

[7] It is possible to obtain these results noting that $-n_0 C_H(q)$ is the *cumulant generating function* of the stochastic force $\mathbf{F}$, which is nonanalytic at $q=0$ and with only one continuous derivative.

$$\widetilde{P}(q) = \int_{-\infty}^{\infty} dx_0 p(x_0)\left[\prod_{n\neq 0}^{-\infty,+\infty}\int_{-\infty}^{\infty} ds p(s + x_0 - n\ell)\exp\left(\frac{iqs}{|s|^{\alpha+1}}\right)\right]. \tag{40}$$

Through an exact analysis [26] of the small-q behavior of Eq. (40), one can distinguish essentially two classes of SL for what concerns the large F behavior of $P(F)$.

(i) First class: No particle can be found arbitrarily close to any other particle, i.e., the supports of $p(u)$ and of $p(u-n\ell)$, respectively, for all integer $n\neq 0$, have an empty overlap. Specifically, this is the case if $\exists\Delta\in(0,\ell/2)$ such that $p(u)=0$ for $|u|>\Delta$, i.e., when the support of $p(u)$ is all contained in one elementary lattice cell. In this case, $P(F)$ vanishes at large F for all values $\alpha>0$ faster than any negative power of F and all the moments $\langle F^n\rangle$ are finite for any $n>0$. If, moreover, $p(u)=p(-u)$, $P(F)$ is not far from a Gaussian with zero mean, even though there are deviations from pure Gaussianity depending on the exact shape of $p(u)$. The finite variance of F in this case is given by

$$\frac{\langle F^2\rangle}{2} = \sum_{l=1}^{+\infty}\sum_{n=1}^{+\infty}[\langle\langle(u-x_0+n\ell)^{-\alpha}\rangle_u\langle(u-x_0+l\ell)^{-\alpha}\rangle_u\rangle_{x_0} - \langle\langle(u-x_0+n\ell)^{-\alpha}\rangle_u\langle(u+x_0+l\ell)^{-\alpha}\rangle_u\rangle_{x_0}] + \sum_{n=1}^{+\infty}\langle\langle(u-x_0+n\ell)^{-2\alpha}\rangle_u - \langle(u-x_0+n\ell)^{-\alpha}\rangle_u^2\rangle_{x_0}, \tag{41}$$

where simply $\langle(\cdots)\rangle_u=\int_{-\infty}^{+\infty}du p(u)(\cdots)$ and $\langle(\cdots)\rangle_{x_0}=\int_{-\infty}^{+\infty}dx_0 p(x_0)(\cdots)$. Thus the force variance is composed of two distinct contributions: (i) the double sum, which is determined basically by the fluctuations created by the stochastic displacement x_0 of the particle initially at the origin on which we evaluate the force (in this term the average over u, i.e., over the sources, plays the role of a smoothing); (ii) the single sum, which is instead mainly due to the fluctuations in the displacements u of all the sources of the force (in this term only the average over x_0 is a smoothing operation). This coincides qualitatively with what we have seen in Eq. (30) for the three-dimensional case with an approximate calculation. The analogy with Eq. (30) can be made stronger with the hypothesis of small displacements, i.e., $\overline{u^2}/\ell^2\ll 1$, where, in analogy with 3D, we have called $\overline{u^2}=\int_{-\infty}^{+\infty}du p(u)u^2$. Keeping only terms up to the second order in the random displacements, it is simple to show that Eq. (41) can be rewritten as

$$\langle F^2\rangle \simeq \frac{2\alpha^2\overline{u^2}}{\ell^{2(\alpha+1)}}[2\zeta^2(\alpha+1)+\zeta(2\alpha+2)], \tag{42}$$

where $\zeta(x)$ is the Riemann zeta function. In this expression, the first contribution comes from the fluctuations of the position of the particle on which we calculate the force, and the latter comes from the fluctuations of the positions of the sources. In particular, by writing

$$\zeta(2\alpha+2) = \sum_{n=1}^{\infty}\frac{1}{n^{2\alpha+2}},$$

it is simple to show that the nth term in the sum gives the relative contribution of the nth NN lattice sites of the origin to the force for $\overline{u^2}/\ell^2\ll 1$.

(ii) Second class: At least one pair of particles (and therefore an infinite number of pairs) can be found arbitrarily close to one each other, i.e., the supports of $p(u)$ and of $p(u-na)$, respectively, for at least an integer $n\neq 0$, have a finite intersection. The simplest case in this class is when $\exists\epsilon>0$ such that $p(u)>0$ for all $|u|<\ell/2+\epsilon$. In this case, it is possible to show [26] that the large-F tail of $P(F)$ is proportional to $F^{-1-1/\alpha}$. More precisely,

$$P(F) \simeq BF^{-1-1/\alpha} \quad \text{for } F\to\infty \tag{43}$$

with

$$B = \frac{1}{\alpha}\int_{-\infty}^{+\infty}dx_0 p(x_0)\sum_{-2u^*/\ell<n<2u^*/\ell}^{n\neq 0}p(x_0-n\ell), \tag{44}$$

where u^* is such that $p(u)>0$ for $u<u^*$ and vanishes for $u>u^*$ [and $u^*=+\infty$ for $p(u)$ with unlimited support]. Note that the large-F exponent of $P(F)$ is independent of the details of $p(u)$. Moreover, it coincides with the exponent characterizing the large-F tail of the 1D Levy PDF found for a 1D Poisson particle distribution,[8] i.e., the 1D analog of the Holtzmark distribution seen in the previous section. In this case, the amplitude B of the tail in the SL is $\ell\int_{-\infty}^{+\infty}dx_0 p(x_0)\sum_{-2u^*/\ell<n<2u^*/\ell}^{n\neq 0}p(x_0-n\ell)<1$ times smaller than the amplitude of the Poisson case, which is simply $1/(\alpha\ell)$ [26]. We note also that, if $u^*\gg\ell$ and $p(u)$ is smooth (i.e., approximately constant) on the length scale ℓ, we can approximate Eq. (44) with

$$B = \frac{1-p(0)\ell}{\alpha\ell}. \tag{45}$$

This last approximated expression is again independent of the details of $p(u)$ for $u\neq 0$.

Intermediate cases between (i), rapidly decreasing $P(F)$, and (ii), Holtzmark-like power law tail of $P(F)$, are possible only if displacements are permitted exactly up to $|u|=\ell/2$ but not beyond this value. In this case, depending on how $p(u)$ behaves in the neighborhood of $u=\pm\ell/2$, one can have intermediate large-F tail behaviors of $P(F)$, e.g., a power-law decreasing faster than $F^{-1-1/\alpha}$.

The results just outlined coincide qualitatively with those we will now find using an approximate generalization of Chandrasekhar's approach to the case of a three-dimensional SL. We will see that this method gives accurate predictions on the large-F behavior of $P(\mathbf{F})$ for all $p(\mathbf{u})$, even though the accuracy for the amplitude of this tail is good only in the limit of sufficiently large displacements.

[8] It is simply found by rephrasing the Chandrasekhar method to the 1D case.

VII. APPROXIMATE CHANDRASEKHAR APPROACH TO $P(\mathbf{F})$ IN THE 3D SL

We extend here the formalism developed in [1], and briefly presented in Sec. VI A, in a similar way to what has been done in [9] for particle distributions generated by a Gauss-Poisson process. As shown in this latter paper, for spatially correlated point processes in which each point has the same mass, it is possible to introduce an approximated PDF for the gravitational (or Coulomb) force felt by each particle (identical charge) of the system and due to all the others. The approximation consists in using only the information carried by one- and two-point correlation functions of the number density field.

Let us consider the general problem of a statistically homogeneous particle distribution in a cubic volume of side L (and hence of volume $V=L^3$) and mean number density n_0 (with $L \gg n_0^{-1/3}$). As usual, the microscopic density is given by Eq. (1) with $m=1$. In analogy with Sec. VI A, we study here the PDF of the total gravitational force on the particle at $\mathbf{x}_0$ due to the other N points in the system in the limit $V \to \infty$ with $N/V=n_0>0$ fixed (taking, as explained above, the limit symmetrically with respect to the point $\mathbf{x}_0$).[9] For simplicity, let us take a coordinate system such that $\mathbf{x}_0$ coincides with the origin. The other N particles occupy the positions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$, respectively. The force acting on the probe particle at the origin is

$$\mathbf{F} = \sum_{i=1}^{N} \frac{\mathbf{x}_i}{x_i^3}. \tag{46}$$

Let $p_c^{(N)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$ be the conditional joint PDF of the positions of the N other particles with respect to the probe at the origin. The approximation we use consists essentially in assuming the validity of the following factorization:

$$p_c^{(N)}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) = \prod_{i=1}^{N} p_c(\mathbf{x}_i), \tag{47}$$

where $p_c(\mathbf{x})$ is the conditional PDF of the position $\mathbf{x}$ of a given particle with the condition that the origin of coordinates is occupied by another particle of the system. We thus approximate the system seen by the particle at the origin with an inhomogeneous Poisson particle distribution with space-dependent average density proportional to $p_c(\mathbf{x})$, insetad of the simple $1/V$ used above for the homogeneous Poisson case. This hypothesis works well in the Gauss-Poisson case [9], and we expect it not to be bad for any particle distribution that does not differ too much from a Poisson one, i.e., with a two-point correlation function $\xi(\mathbf{x})$ that is short-range (i.e., integrable) and with a small amplitude. This means that in our case of a SL, we expect that the approximation will give a quantitatively good estimate of $P(\mathbf{F})$ only when the typical displacement of a particle starts to be of the order of the lattice cell size. In fact, it is only in this case that the lattice Bragg peaks contribution to the PS of Eq. (7) is strongly reduced and, consequently, the amplitude of $\xi(\mathbf{x})$ is small enough. However, we will see that even for smaller displacements, when the force variance is finite, this approximation gives good quantitative predictions about the large-F tail of $P(\mathbf{F})$, even though the value of the force variance is accurate only for the case in which the largest permitted displacement starts to approach the cell boundary. This means that when instead the maximal permitted displacement is much smaller than the lattice cell size, we keep the qualitative results we present in this section but use for the variance of $\mathbf{F}$ Eqs. (28) and (30).

Directly from the definition of the two-point correlation function, we have

$$p_c(\mathbf{x}) = A[1 + \xi(\mathbf{x})], \tag{48}$$

where $A \simeq 1/V$ is the normalization constant, and $\xi(\mathbf{x})$ for any stationary point process is defined as the off-diagonal part of the covariance function $\tilde{\xi}(\mathbf{x})$ [see Eq. (4)].

Making these hypotheses, and following the same steps written above for the homogeneous Poisson case, it is possible to write [10], in the infinite volume limit with n_0 fixed, the PDF of $\mathbf{F}$ as

$$P(\mathbf{F}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{F}} \exp\left\{ -n_0 \int d^3x [1+\xi(\mathbf{x})] \times (1 - e^{-i\mathbf{q}\cdot\mathbf{x}/x^3}) \right\}. \tag{49}$$

It will be useful to rewrite this in the form

$$P(\mathbf{F}) = \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{F}} \exp\left[-n_0 C_H(q) - n_0 \int d^3x \xi(\mathbf{x}) (1 - e^{-i\mathbf{q}\cdot\mathbf{x}/x^3}) \right], \tag{50}$$

where $C_H(q)=4(2\pi q)^{3/2}/15$, with the multiplicative factor $-n_0$, is the cumulant generating function for $\mathbf{F}$ in the homogeneous Poisson case already considered in Sec. VI A [see Eq. (34)]. The function

$$A(\mathbf{q}) \equiv \int d^3F e^{-i\mathbf{q}\cdot\mathbf{F}} P(\mathbf{F}) = \exp\left\{ -n_0 \int d^3x [1+\xi(\mathbf{x})](1 - e^{-i\mathbf{q}\cdot\mathbf{x}/x^3}) \right\} \tag{51}$$

is the characteristic function of the stochastic force $\mathbf{F}$. We recall that the function

$$\mathcal{G}(\mathbf{q}) = \ln A(\mathbf{q}) = -n_0 \int d^3x [1+\xi(\mathbf{x})](1 - e^{-i\mathbf{q}\cdot\mathbf{x}/x^3}) \tag{52}$$

is the *cumulant generating function* of the stochastic field $\mathbf{F}$ [27]. The cumulants (i.e., the connected parts of the moments) of $\mathbf{F}$ can be directly calculated by taking the derivatives of this function at $q=0$. Therefore, the small-q behavior

[9]Because of the stochastic nature of the point process, N can deviate from $n_0 V$ by a quantity growing slower than V, which thus does not affect the results we present, which are given in the infinite volume limit.

of $\mathcal{G}(\mathbf{q})$ describes the large-F properties of $P(\mathbf{F})$. Since in Eq. (51) the small-q region corresponds to the small-x region of $\xi(\mathbf{x})$, we can say roughly that the large-F behavior of $P(\mathbf{F})$ is basically determined by the small separation properties of the particle distribution [and therefore on the small x behavior of $\xi(\mathbf{x})$]. We have already considered this aspect both for the Chandrasekhar method for the homogeneous Poisson particle distribution and for the exact results in 1D. For the homogeneous Poisson case, the off-diagonal part of the reduced correlation function is $\xi(\mathbf{x})\equiv 0$, and Eq. (51) becomes consistently Eq. (34), which implies Eq. (36a).

A. The large-F behavior of $P(\mathbf{F})$

In order to simplify the calculations which follow, we make the assumption that $\xi(\mathbf{x})=\xi(x)$ even though, rigorously speaking, this is not the case for our SL because of the underlying lattice symmetry even when $p(\mathbf{u})$ depends only on $u=|\mathbf{u}|$. For the SL this can be seen as an approximation consisting in substituting $\xi(\mathbf{x})$ in Eq. (50) with its angular average. Assuming $\xi(\mathbf{x})=\xi(x)$ implies that $A(\mathbf{q})$ and $P(\mathbf{F})$ depend, respectively, only on q and F, and Eq. (52) can be rewritten as

$$\begin{aligned}\mathcal{G}(\mathbf{q}) &= -4\pi n_0\int_0^\infty dx x^2[1+\xi(x)]\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right]\\ &= -n_0\Bigg\{C_H(q)+4\pi\int_0^\infty dx x^2\xi(x)\\ &\quad\times\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right]\Bigg\}. \qquad (53)\end{aligned}$$

Let us now analyze how the shape of $\xi(x)$ determines the large-F tail of $P(\mathbf{F})$. To do this, the fundamental step is to study the small-q behavior of $\mathcal{G}(\mathbf{q})$. In this respect, we distinguish below three cases for the choice of $p(\mathbf{u})$ with a continuous and convex support: (Sec. VII A 1, large displacements) it permits, at least in some directions, displacements of particles beyond the border of their elementary lattice cells, allowing, in this way, different particles to be found arbitrarily close to one each other; (Sec. VII A 2, marginal displacements) it permits, at least in some directions, displacements exactly up to the border of the elementary lattice cell and in no direction beyond it, allowing different particles to be found arbitrarily close to one each other, but only marginally; (Sec. VII A 3, small displacements) its support is all contained in an internal region of the elementary lattice cell so that there is a finite lower bound on the distance between any two particles.

As has been noted in the discussion above about the accuracy of the approximation (47) and (48), we expect to obtain better and better approximations for $P(\mathbf{F})$ the larger the typical particle displacement is.

In all three cases, it is important to note that $\mathcal{G}(0)=0$. Moreover, as a SL is "superhomogeneous," $\int d^3x\xi(\mathbf{x})=-n_0=-\ell^{-3}$. We now treat one by one the above three cases.

1. Large displacements

In this case, it is simple to show that $\xi(0)>-1$ and finite. In fact, the average conditional density $n_0[1+\xi(\mathbf{x})]$ has to converge to a positive constant for $x\to 0$, as the large displacements permit couples of particles to be found arbitrarily close to one another. This is sufficient to show (see Appendix A) that, up to the leading term at small q, we have

$$\mathcal{G}(\mathbf{q}) \simeq -\frac{4}{15}n_0[1+\xi(0)](2\pi q)^{3/2},$$

which is of the same order in q of $C_H(q)$. By recalling that $A(\mathbf{q})=\exp[\mathcal{G}(\mathbf{q})]$ and performing the Fourier transform (50), we can simply derive at large F that

$$P(\mathbf{F}) \simeq [1+\xi(0)]P_H(\mathbf{F}),$$

which can be rewritten in terms of the PDF of $F=|\mathbf{F}|$, $W(F)=4\pi F^2P(\mathbf{F})$, as

$$W(F) \simeq [1+\xi(0)]W_H(F) \simeq [1+\xi(0)]\frac{15}{8}\sqrt{\frac{2}{\pi}}F_*^{3/2}F^{-5/2}, \qquad (54)$$

where we have used Eq. (36b). This means that in this case $P(\mathbf{F})$ presents the same large force scaling behavior as that of the Holtzmark distribution, but with an amplitude greater by a factor $1+\xi(0)$. Given that $n_0[1+\xi(0)]$ is the average conditional density of other particles at zero distance, it is simple to show that for a SL one can write

$$1+\xi(0) = \frac{1}{n_0}\sum_{\mathbf{R}\neq\mathbf{0}}\int d^3u p(\mathbf{u})p(\mathbf{u}-\mathbf{R}),$$

from which one finds the 3D analog of Eq. (44). Note that for most choices of $p(\mathbf{u})$ one has also $\xi(0)<0$ (i.e., as explained in Sec. II, the system is negatively correlated at small scales). Therefore, the amplitude of the tail will usually be reduced with respect to that in the Holtzmark case. In any case, as for the Holtzmark distribution, all the generalized moments $\langle F^\eta\rangle$ diverge for $\eta\geqslant 3/2$.

2. Marginal displacements

In this case $\xi(x)\simeq[-1+Bx^\beta]$ with $B,\beta>0$ at small x. In fact, as displacements are permitted up to exactly the cell boundary, the probability of finding a particle at distance x from another fixed particle must vanish continuously for $x\to 0$. By studying the small-scale behavior of $\mathcal{G}(\mathbf{q})$ (see Appendix A), we can conclude that for $q\to 0$ we have the following.

(i) $\mathcal{G}(\mathbf{q})\sim -q^{(3+\beta)/2}$ if $0<\beta<1$, which implies $W(F)\sim F^{-(5+\beta)/2}$ at large F. In this case, the variance $\langle F^2\rangle$ is thus divergent, but slower than in the Holtzmark case.

(ii) $\mathcal{G}(\mathbf{q})\sim q^2\log q$ if $\beta=1$, implying $W(F)\sim F^{-3}$ (giving a logarithmically divergent variance $\langle F^2\rangle$).

(iii) $\mathcal{G}(\mathbf{q})\sim -q^2+o(q^2)$ for $\beta>1$, where $o(q^2)$ is a power vanishing faster than 2 and including the main singular part of this small-q expansion proportional to $q^{(3+\beta)/2}$ (with logarithmic corrections for β integer larger than 1). This implies again $W(F) \sim F^{-(5+\beta)/2}$ at large F (giving a finite variance $\langle F^2 \rangle$).

Summarizing, we can say that, in general, $W(F) \sim F^{-(5+\beta)/2}$ with $\beta > 0$ at large F, implying that all the moments $\langle F^\eta \rangle$ diverge for $\eta \geq (3+\beta)/2$.

3. Small displacements

In this last case, displacements are permitted up to a distance $\Delta < \ell/2$. Consequently, there will be a positive minimal distance $x^* = \ell - 2\Delta$ between any two particles. This implies that $\xi(x) = -1$ identically for $x < x^*$, as around any particle there is a minimal empty region of radius x^*. Therefore, Eq. (53) can be rewritten as

$$\mathcal{G}(\mathbf{q}) = -4\pi n_0 \int_{x^*}^{\infty} dx x^2 [1 + \xi(x)] \left[1 - \frac{x^2}{q} \sin\left(\frac{q}{x^2} \right) \right]. \tag{55}$$

The large-F behavior of Eq. (50) is essentially determined by the small-q behavior of this function. Since $x^* > 0$, and $\xi(x) \to 0$ for $x \to \infty$, one can verify easily that the integral in Eq. (55) can be expanded in Taylor series to all orders in q with finite coefficients, i.e., $\mathcal{G}(\mathbf{q})$ and, consequently, $A(\mathbf{q})$ are analytic functions of q. From Fourier transform theorems [28], one can then infer that $P(\mathbf{F})$ vanishes at large F faster than any negative power of F, i.e., $W(F)$ has all moments $\langle F^n \rangle$ finite. To leading order in q, we can write

$$\int_{x^*}^{\infty} dx x^2 [1 + \xi(x)] \left[1 - \frac{x^2}{q} \sin\left(\frac{q}{x^2} \right) \right] = \frac{q^2}{6} \int_{x^*}^{\infty} dx \frac{1 + \xi(x)}{x^2} + O(q^4). \tag{56}$$

Therefore, at small q the characteristic function $A(\mathbf{q})$ has the following behavior:

$$A(\mathbf{q}) = 1 - \frac{q^2 \sigma_F^2}{6} + O(q^4), \tag{57}$$

where

$$\sigma_F^2 = 4\pi n_0 \int_{x^*}^{\infty} dx \frac{1 + \xi(x)}{x^2} \tag{58}$$

is the approximation we obtain with this method for the variance $\langle F^2 \rangle$ of the force $\mathbf{F}$. It is important to stress that this formula for σ_F^2 is expected to apply to our SL only when Δ/ℓ approaches sufficiently the value $1/2$, i.e., when displacements are large enough to make the amplitude of $\xi(x)$ small. For smaller values of Δ, Eqs. (28) and (30) have instead to be applied to calculate $\langle F^2 \rangle$.

Note that the 3D isotropic (i.e., monovariate) and uncorrelated Gaussian PDF reads

$$P_G(\mathbf{F}) = \left[\frac{3}{2\pi \sigma_F^2} \right]^{3/2} \exp\left(-\frac{3F^2}{2\sigma_F^2} \right) \tag{59}$$

and has a characteristic function

$$A_G(\mathbf{q}) = \exp\left(-\frac{q^2 \sigma_F^2}{6} \right), \tag{60}$$

which has the same second-order small-q expansion as that of Eq. (57) [i.e., the same lowest-order cumulant-generating function $\mathcal{G}(\mathbf{q})$]. Therefore, we can say that in our case, if Δ is not much smaller than $\ell/2$, $P(\mathbf{F})$ is approximately given by the 3D Gaussian (59) where $\langle F^2 \rangle$ is approximated by σ_F^2, which is given by the covariance function via Eq. (58). Note, however, that Gaussianity is only approximate. In fact, terms of order equal to or higher than q^4 from the expansion of Eq. (55) are in general not vanishing, differently from the case of Eq. (59) where the cumulant-generating function is a simple quadratic function of $\mathbf{q}$. Instead, the fourth-order term of the Taylor expansion of Eq. (55) can be written in general as $4\pi n_0 c_4 q^4$ [the quantity $(\pi/6) n_0 c_4$ is the fourth *cumulant* of $\mathbf{F}$] with

$$c_4 = \frac{1}{5!} \int_{x^*}^{\infty} dx \frac{1 + \xi(x)}{x^6}.$$

In order to evaluate how large is the deviation from pure Gaussianity due to the fourth- (and higher-order) cumulant term, one has to compare $4\pi n_0 c_4$ with $\sigma_F^4/72$. In fact, the fourth-order term of the Taylor expansion of the pure Gaussian $A_G(\mathbf{q})$ [Eq. (60)] is $(\sigma_F^4/72) q^4$, while in our case the fourth-order term of $A(\mathbf{q})$ is $(\sigma_F^4/72 + 4\pi n_0 c_4) q^4$. Therefore, the quantity

$$\lambda_{\rm NG} = \frac{288\pi n_0 c_4}{\sigma_F^4} = \frac{3}{20\pi n_0} \frac{\int_{x^*}^{\infty} dx [1 + \xi(x)]/x^6}{\left[\int_{x^*}^{\infty} dx [1 + \xi(x)]/x^2 \right]^2} \tag{61}$$

is a good measure of the degree of the *non-Gaussianity* of $P(\mathbf{F})$. The quantity $\lambda_{\rm NG}$ is called in probability theory the *kurtosis excess* [29]. It measures the importance of the large-F tail with respect to the Gaussian case with the same variance. When $\lambda_{\rm NG} \ll 1$ (i.e., $2\Delta/\ell \ll 1$), we can say that the deviation of $P(\mathbf{F})$ from the Gaussian $P_G(\mathbf{F})$ in Eq. (59) is small, while, on the other hand, if instead $\lambda_{\rm NG} \simeq 1$, the deviation starts to be appreciable and the large-F tail of $P(\mathbf{F})$ starts to be considerably fatter than that of $P_H(\mathbf{F})$, and finally for $\lambda_{\rm NG} \gg 1$ (i.e., for $\Delta \simeq \ell/2$) the Gaussian approximation is inappropriate and $P(\mathbf{F})$ starts to develop the power-law tail described above for the cases of large and marginal displacements. This deviation from Gaussianity (see below) clearly increases with Δ and in general diverges when Δ approaches $\ell/2$: in fact, for this value all the moments higher than a given value diverge. Therefore, for $2\Delta \to \ell^-$ we expect to see large discrepancies of $P(\mathbf{F})$ with respect to Eq. (59).

In general, for small $(x - x^*) > 0$ the covariance $\xi(x)$ in the present case behaves as $[1 + \xi(x)] = B(x - x^*)^\beta$ with positive B and β depending on $p(\mathbf{u})$. By changing the value of β [i.e., in our case of a SL, by changing the scaling behavior of $p(\mathbf{u})$ in the neighborhood of $u = \Delta$], we can have or not a diverging behavior of σ_F^2 and c_4 when $x^* \to 0^+$, that is, when $\Delta \to (\ell/2)^-$, and displacements are permitted up to exactly the lattice cell boundaries. However, in general we can say that the deviation from pure Gaussianity given by λ_{NG} increases with Δ and diverges at $\Delta=\ell/2$ for $\beta<5$.

For what concerns the approximate variance σ_F^2, it is simple to show that if (i) $\beta<1$, this quantity diverges as $x^{*\beta-1}\sim(\ell-2\Delta)^{\beta-1}$, if (ii) $\beta=1$, it diverges as $-\log(\ell-2\Delta)$ (as we have already seen above in the case in which $\Delta=\ell/2$ exactly), and if (iii) $\beta>1$, it converges to a finite value. In a similar way it is simple to see that c_4 diverges as $(\ell-2\Delta)^{\beta-5}$ for $\beta<5$ [implying a large-F tail slower than $1/F^7$, for $P(\mathbf{F})$], logarithmically for $\beta=5$, and converges otherwise.

Finally, Gaussianity [i.e., a PDF given by Eq. (59)] is almost exact when $\Delta\ll\ell/2$.[10] However in this case the approximation given by Eq. (58) for the variance $\langle F^2\rangle$ of $\mathbf{F}$ in the SL is not a good one. In fact, in this case substituting such a SL with inhomogeneous Poisson particle distribution with a radial density around the origin (where we calculate the force) equal to the average conditional density of the SL is a bad approximation as $\xi(x)$ acquires large values around the Bragg peaks. Nevertheless, following also the results in $d=1$ presented above, in this case we can say again that $\mathbf{F}$ is approximately a 3D Gaussian variable [i.e., with a PDF given by Eq. (59)], but with $\langle F^2\rangle$ given by Eq. (28).

B. Small-F behavior of $P(\mathbf{F})$

In order to find the small-F behavior of $P(\mathbf{F})$, first of all we note [see the first formula of Eq. (36a)] that in the homogeneous Poisson case

$$P_H(0)=4\pi\int_0^\infty dqq^2\exp[-n_0C_H(q)]$$

is finite, i.e., $W_H(F)\sim F^2$. In our case, from Eqs. (50) and (53), $P(\mathbf{F})$ for $\mathbf{F}=\mathbf{0}$ is given by

$$P(0)=\int d^3qA(\mathbf{q})=4\pi\int_0^\infty dqq^2\exp\left\{-n_0C_H(q)-4\pi n_0\int_0^\infty dxx^2\xi(x)\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right]\right\}. \quad (62)$$

It is simple to verify that for any possible covariance functions $\xi(x)$ the quantity $P(0)$ stays finite, i.e., again $W(F)\sim F^2$. Roughly speaking, the more the particle distribution shows anticorrelations, the larger will be the value $P(0)$, i.e., the larger will be the probability of observing a small value of F. On the contrary, the larger the positive correlations, the smaller the value of $P(0)$. In particular, in our case of a randomly perturbed lattice the system is *superhomogeneous*, that is, $\int d^3x\xi(\mathbf{x})=-1$, and therefore negative density-density correlations are always more important than positive correlations. This means that in general, given the structure of Eq. (62) (and in particular the fact that $\sin t/t<1$ for any $t>0$), $P(0)>P_H(0)$. Only in the limit of random displacements in the whole system volume do we have $P(0)\to P_H(0)$. Moreover, in general, the smaller the typical displacements are, the larger will be the contribution of anticorrelations to Eq. (62) and then the larger $P(0)$.

[10]This can be seen more rigorously noting that for this range of Δ, the linear expansion (28) is valid. Therefore, each component of the force $\mathbf{F}$ can be seen as the sum of independent random variables with finite variance and satisfying the *Lindeberg* condition [30]. This allows us to apply the central limit theorem with good approximation to each component of $\mathbf{F}$, which, consequently, becomes a well-defined Gaussian variable in the infinite volume limit.

VIII. COMPARISON WITH NUMERICAL SIMULATIONS

In this section, we compare our theoretical predictions for the statistics of $\mathbf{F}$ (specifically, the variance and the global PDF) we have given in the previous sections with numerical results for the same quantities obtained directly by computer simulations of the SL particle distribution with given $p(\mathbf{u})$. The paradigmatic example, on which we concentrate our numerical analysis, is given by the case in which the three components u_i with $i=1,2,3$ along the three orthogonal axes of the displacement $\mathbf{u}$ applied to the generic particle are statistically independent and uniformly distributed in a symmetric interval, i.e.,

$$p(\mathbf{u})=\prod_{i=1}^3 f(u_i), \quad (63)$$

with

$$f(u_i)=\begin{cases}\dfrac{1}{2\Delta} & \text{for } |u_i|\le\Delta\\[2mm] 0 & \text{otherwise.}\end{cases} \quad (64)$$

In this case, the average quadratic displacement is

$$\overline{u^2}=\frac{3}{2\Delta}\int_{-\Delta}^{+\Delta}x^2f(x)dx=\Delta^2$$

or $\overline{u^2/\ell^2}=\delta^2$ in normalized units $\delta=\Delta/\ell$. The FT of $p(\mathbf{u})$, i.e., the characteristic function of the random displacements, is simply given by

$$\hat{p}(\mathbf{k})=\mathcal{F}[p(\mathbf{u})]=\prod_{j=1}^3\frac{\sin(k_j\Delta)}{k_j\Delta},$$

where k_j is the jth component of $\mathbf{k}$. By using Eq. (7), it is then simple to verify that the PS $\mathcal{P}(\mathbf{k})$ of the SL is given by

$$\mathcal{P}(\mathbf{k})=\ell^3\left[1-\prod_{j=1}^3\frac{\sin^2(k_j\Delta)}{(k_j\Delta)^2}\right]+(2\pi)^3\sum_{\mathbf{H}\neq 0}\left[\prod_{j=1}^3\frac{\sin^2(H_j\Delta)}{(H_j\Delta)^2}\right]\delta(\mathbf{k}-\mathbf{H}). \quad (65)$$

This expression reduces at small k to [19,20]

$$\mathcal{P}(\mathbf{k})=\frac{1}{3}\ell^3\Delta^2k^2=\frac{1}{3}\ell^5\delta^2k^2, \quad (66)$$

which depends only on $k=|\mathbf{k}|$, i.e., mass (or number) fluctuations are statistically isotropic on large scales even though the SL is not because of the underlying lattice symmetry. In Fig. 2, the PS of the 1D analog of such a SL with $\delta < 1/2$ is given. In this figure, both the k^2 scaling behavior at small k and the modulation of the Bragg peaks of the initial lattice by the factor $|\hat{p}(k)|^2$ are clear.

For our particular choice of $p(\mathbf{u})$, it is possible to calculate the inverse FT of Eq. (65) to obtain, through Eq. (4), the off-diagonal covariance function $\xi(\mathbf{x})$ for all values of Δ. Let us call n the integer part of the ratio $4\Delta/\ell$. We can write

$$1+\xi(\mathbf{x}) = \left(\frac{\ell}{2\Delta}\right)^3 \prod_{j=1}^{3}\left(\frac{|x_j|}{2\Delta}-1\right)\theta(2\Delta-|x_j|) + \prod_{j=1}^{3}\left[1-\left(\frac{\Delta'}{\Delta}\right)^2 + \frac{\ell}{(2\Delta)^2}\sum_{m=-\infty}^{+\infty}(2\Delta'-|x_j-m\ell|)\theta(2\Delta'-|x_j-m\ell|)\right], \tag{67}$$

where $\theta(x)$ is the usual Heaviside step function, and $2\Delta' = (2\Delta - n\ell/2)$ if n is even or zero and $2\Delta' = [(n+1)\ell/2 - 2\Delta]$ if n is odd. It is important to notice that $0 \leq 2\Delta' < \ell/2$ for all Δ values. The function $\xi(\mathbf{x})$ given by Eq. (67) has in general a very complicated oscillating form, with the exception, as shown below, of the case in which $\Delta = m\ell/2$ exactly with m integer. However, for a generic Δ we can say that it is continuous and is composed of two different contributions. The former is given by the first product of Eq. (67) and comes from the continuous (i.e., purely stochastic) part of the PS, and the latter, given by the second product that is a lattice periodic function, comes from the modulated Bragg peaks part of the PS. For all the choices $\Delta = m\ell/2$ with integer m, the "Bragg peaks contribution" exactly vanishes leaving only the first stochastic contribution: this means that for these values of Δ, the system becomes statistically translationally invariant. Note that for $\Delta < \ell/2$ we have $\xi(\mathbf{x}) = -1$ identically in the cube of side $|x_i| \leq (\ell - 2\Delta)$ for all $i = 1,2,3$, meaning that the probability of finding a particle in such a cube centered around any other particle is strictly zero for these values of Δ.

We can now draw the following general conclusions for all the possible choices of $\delta = \Delta/\ell$. For simplicity, we start with the limiting case $\delta = 1/2$, and then we analyze, respectively, the cases $\delta < 1/2$ and $\delta > 1/2$.

A. Large-F prediction for $\Delta = \ell/2$

For $\delta = 1/2$, it is straightforward to find the exact result

$$\xi(\mathbf{x}) = -\prod_{j=1}^{3}(1-|x_j|/\ell)\,\theta(\ell-|x_j|). \tag{68}$$

Thus $\xi(\mathbf{x})$ is in this case nonvanishing, and negative, only in the cube $-\ell < x_j < \ell$ for all $j = 1,2,3$. As mentioned above, in this case the lattice Bragg peaks of Eq. (65) are completely erased by the displacements and the system is statistically invariant under translations. Expression (68) at $x \ll \ell$ gives

$$\xi(\mathbf{x}) \simeq -1 + \sum_{j=1}^{3}\frac{|x_j|}{\ell}. \tag{69}$$

Therefore, the SL falls in the class of Sec. VII A 2 with $\beta = 1$, i.e., with $P(\mathbf{F}) \sim F^{-5}$ at large F [or equivalently $W(F) \sim F^{-3}$] and, from Eq. (58), logarithmically diverging variance σ_F^2. In order to have a more quantitative description of this case, we mimic the anisotropic Eq. (68) with the following isotropic $\xi(x)$:

$$\xi'(x) = \left[-1+\left(\frac{\pi}{6}\right)^{1/3}\frac{x}{\ell}\right]\theta\left[\left(\frac{6}{\pi}\right)^{1/3}\ell - x\right], \tag{70}$$

i.e., an isotropic function with a linearly increasing behavior, similar to the one in Eq. (69), from $\xi' = -1$ at $x = 0$ to $\xi' = 0$ at the border and outside the sphere centered at $x = 0$ with the same volume $(2\ell)^3$ [i.e., radius $R = (6/\pi)^{1/3}\ell$] as that of the cubic region in which the function $\xi(\mathbf{x})$ in Eq. (68) increases from -1 to 0. By using this expression in Eq. (53), and for $q \ll \ell^2$, we obtain

$$A(\mathbf{q}) \simeq \exp[2(\gamma q)^2 \ln(\gamma q)] = \epsilon(q)^{\epsilon(q)},$$

with $\gamma = (\pi/6)^{2/3}\ell^{-2}$ and $\epsilon(q) = (\gamma q)^2$. By studying the inverse FT leading from $A(\mathbf{q})$ to $P(\mathbf{F})$, and therefore to $W(F)$, we obtain at large F

$$W(F) \simeq 2\pi\left(\frac{\pi}{6}\right)^{1/3}\ell^{-4}F^{-3}. \tag{71}$$

B. Large-F prediction for $\Delta < \ell/2$

For $\delta < 1/2$, we can say that the SL falls in the class of Sec. VII A 3, with $\beta = 1$. It is simple to verify from Eq. (67) that $\xi(\mathbf{x}) = -1$ identically in the cube $|x_i| \leq (\ell - 2\Delta)$ with $i = 1,2,3$. As written in the previous section, in this case $P(\mathbf{F})$ is expected to be rapidly decreasing at large F and with finite average quadratic force $\sigma_F^2 = \langle F^2 \rangle$ as all the higher-order moments.

More precisely, for $\delta \ll 1/2$, $P(\mathbf{F})$ is expected to be given, to a good approximation, by Eq. (59) (i.e., Gaussian with a small kurtosis excess) with a variance σ_F^2 given by Eq. (30), where $\overline{u^2} = \Delta^2$. Instead for δ approaching 1/2, i.e., $(1-2\delta) \ll 1$, we expect σ_F^2 to be given, once a suitable isotropic approximation is introduced, by Eq. (58). A large kurtosis excess λ_{NG} for this range of δ is also expected, implying large deviations from pure Gaussianity. It is particularly interesting to study the diverging behavior of σ_F^2 for $\delta \to (1/2)^-$ using Eq. (58) to test the validity of this approximated analytical result through comparison with measures from numerical simulations. In order to apply Eq. (58) to our SL case, we need an isotropic approximation as good as possible for $\xi(\mathbf{x})$. First of all it is important to note that (i) for $\mathbf{x}$ just outside the cube $|x_i| \leq (\ell - 2\Delta)$ with $i = 1,2,3$, the function $\xi(\mathbf{x})$ grows linearly, (ii) $\xi(\mathbf{x})$ grows up to the surface of the cube $|x_i| \leq 2\Delta \simeq \ell$ with $i = 1,2,3$, (iii) outside the cube $\xi(\mathbf{x})$ is a function with the periodicity of the lattice with amplitude at most of order $(1-2\delta) \ll 1$, and with zero mean on the period (i.e., on the elementary cell). These observations, combined with the same argument leading to Eq. (70), for $\delta=1/2$, permit one to approximate $\xi(\mathbf{x})$ simply with

$$\xi'(x) \simeq \begin{cases} -1 & \text{for } x < x^* \\ \left[\left(\frac{\pi}{6}\right)^{1/3}\frac{(x-x^*)}{2\Delta}-1\right]\theta\left[\left(\frac{6}{\pi}\right)^{1/3}\ell - x\right] & \text{for } x \geq x^*, \end{cases} \tag{72}$$

where $x^*=(6/\pi)^{1/3}(\ell-2\Delta)$ is chosen so that $\xi'(x)=-1$ in the whole sphere around $x=0$ with the same volume $8(\ell-2\Delta)^3$ as the cube where the exact $\xi(\mathbf{x})=-1$ identically, and $(6/\pi)^{1/3}\ell$ is analogously the radius of the sphere with volume $8\ell^3$ [i.e., the volume of the cube around $\mathbf{x}=\mathbf{0}$ outside of which $\xi(\mathbf{x})$ is everywhere small and at most of order $(1-2\delta)$]. Clearly this is a rough approximation to $\xi(\mathbf{x})$. However, we will see that it permits one to predict both the logarithmic divergence in $(1-2\delta)$ of σ_F^2 and its order of magnitude. In fact, by using the function (72) as $\xi(x)$ in Eq. (58), with $n_0=l^{-3}$, we obtain the following logarithmically diverging behavior for $\delta\to(1/2)^-$ of σ_F^2:

$$\sigma_F^2 \simeq -4\pi\left(\frac{\pi}{6}\right)^{1/3}\frac{\ln(1-2\delta)}{2\delta\ell^4}. \tag{73}$$

Moreover, as discussed in the previous section, for values of δ such that $(1-2\delta)\ll 1$ the kurtosis excess λ_{NG} is expected to be large, implying a large deviation from Gaussianity of $P(\mathbf{F})$. This can be seen by using Eq. (72) in Eq. (61), which gives for the present case

$$\lambda_{NG} \simeq \frac{\delta}{400[\ln(1-2\delta)]^2}(1-2\delta)^{-4}. \tag{74}$$

It is important to underline that this result is valid when Eq. (72) is valid, i.e., when δ is not too far from the value 1/2. From Eq. (74) one can simply see that $\lambda_{NG}\approx 0.24$ for $\delta=0.4$ and $\lambda_{NG}\approx 1.18$ for $\delta=0.44$. Moreover, beyond this value, λ_{NG} diverges monotonously as $(1-2\delta)^{-4}$ for larger δ. The right interpretation of this result is given directly by the statistical meaning of the *kurtosis*: by increasing δ in this range, $W(F)$ becomes more peaked and with a large-F tail fatter and fatter than the 3D Gaussian distribution (59) with the same variance σ_F^2.

C. Large-F prediction for $\Delta>\ell/2$

Since the approximated Chandrasekhar approach to the SL problem is more and more precise when δ increases, we expect for $\delta>1/2$ a better quantitative agreement between the analytic results and the numerical simulations than in the previous two cases. In this range of displacements one has $\xi(0)>-1$, meaning that on average a particle sees a density of other particles larger than zero at a vanishing separation, as each particle can be found arbitrarily close at least to another particle. This implies that such a SL falls in the class of Sec. VII A 1 with a large-F tail of $W(F)$ with the same scaling as that of the Holtzmark distribution but with a different amplitude depending on the value of $\xi(0)$. This is clear from Eq. (54), which we rewrite here for convenience as

$$W(F)=[1+\xi(0)]W_H(F) \tag{75}$$

at large F, where $W_H(F)$ is given by Eq. (36b). As explained in Sec. VII A, as in the homogeneous Poisson case, the statistically dominant contribution to the force acting on a particle in this ccase comes from its nearest neighbor. From Eq. (67) it is simple to find that

$$1+\xi(0)=\left[1+\frac{\ell}{2\Delta}\left(\frac{\Delta'}{\Delta}\right)-\left(\frac{\Delta'}{\Delta}\right)^2\right]^3-\left(\frac{\ell}{2\Delta}\right)^3. \tag{76}$$

Therefore, depending on the choice of Δ we can have both $[1+\xi(0)]$ smaller or larger than 1, i.e., with a statistical weight for large values of F smaller or larger than in the homogeneous Poisson particle distribution, respectively, depending on whether the probability of finding the nearest neighbor at very small distances is smaller or larger than in the Poisson case. However, as the initial lattice configuration presents negative density-density correlations at small scales, for most of the choices of Δ one has $-1<\xi(0)<0$ and therefore the large-F tail of $W(F)$ has a smaller amplitude than $W_H(F)$. In general, by taking only the largest terms beyond one in Eq. (76), we can say that for large Δ, the large-F ratio of $W(F)/W_H(F)$ approaches unity as $[1-O(\delta^{-2})]$, where $O(x)\sim x$.

D. Small-F predictions

For what concerns the small-F behavior of $W(F)$, we have already seen in the previous section that for all values of δ we have $W(F)\simeq 4\pi P(0)F^2$ with the prefactor $P(0)$ depending on $\xi(x)$. More precisely, for $\delta\ll 1/2$ we have just seen that $P(\mathbf{F})$ coincides to a good approximation with the Gaussian (59). Therefore, one has simply $P(0)=[3/(2\pi\sigma_F^2)]^{3/2}$, where σ_F^2 is given by Eq. (30). For higher values of δ, when the approximate Chandrasekhar approach starts to work, in order to find $P(0)$, one should solve the integral (62) where $\xi(x)$ is some appropriate isotropic approximation of Eq. (67). Clearly this is a task that it is very difficult or impossible to perform analytically. However, for $\delta\geq 1/2$, i.e., when $W(F)$ is power law at large F, one can adopt the following simple method to have a rough approximation for the amplitude of the small-F tail of $W(F)$ and therefore obtain a useful approximation of $W(F)$ for all values of F to be used to evaluate averages of arbitrary functions of F. One assumes the following simple shape for $W(F)$,

$$W(F)=\begin{cases} AF^2 & \text{for } F<F_0 \\ BF^{-\alpha} & \text{for } F\geq F_0, \end{cases} \tag{77}$$

where, respectively, as found above, $\alpha=3$ and $B=2\pi(\pi/6)^{1/3}\ell^{-4}$ for $\delta=1/2$, and $\alpha=5/2$ and $B=2\pi[1+\xi(0)]\ell^{-3}$ [where we have used Eqs. (75) and (36a)] with $\xi(0)$ given by Eq. (76) for $\delta>1/2$. In order to find A and F_0, we impose the following two conditions: (i) small-F and large-F tails take the same value at $F=F_0$, (ii) normalization of $W(F)$, i.e., $\int_0^\infty dFW(F)=1$. The first condition implies

$$AF_0^2 = BF_0^{-\alpha},$$

while the second (normalization) condition gives the equation

$$\frac{A}{3}F_0^3 + \frac{B}{\alpha-1}F_0^{1-\alpha} = 1.$$

These two equations can be solved to give

$$A = BF_0^{-2-\alpha},$$

$$F_0 = \left[\frac{3(\alpha-1)}{B(\alpha+2)}\right]^{1/(1-\alpha)}. \quad (78)$$

In particular, it is important to note that for $\delta > 1/2$ these formulas imply that

$$F_0 \sim [1+\xi(0)]^{2/3},$$

$$A \sim [1+\xi(0)]^{-2}, \quad (79)$$

where again $\xi(0)$ is given by Eq. (76).

E. Comparison with numerical simulations

To test the above analytical results, we have generated numerically several simple cubic SL, with fixed ℓ (for simplicity we have chosen $\ell=1$, i.e., $n_0=1$) and different values of Δ in order to study σ_F^2 and $W(F)$ in a wide range going from $\delta \ll 1/2$ to $\delta > 1/2$. We expect then to see the transition of $P(\mathbf{F})$ from nearly Gaussian for small δ to nearly Holtzmark for large values of δ when the particle distribution approaches the Poisson one. For each chosen value of δ, we have evaluated the PDF $W(F)$ in the following way: for each realization of the SL, the force is evaluated on the "central" particle (i.e., on the particle farthest from the boundaries of the system); then $W(F)$ is evaluated as a normalized histogram over 10^5 realizations. The force $\mathbf{F}$ on the central particle is computed by using the Ewald sum method for lattice sums for the cases $\delta < 0.3$ (i.e., when the SL keeps clear lattice features) in order to make this evaluation faster and precise. For larger values of δ, on the other hand, F is given by the simple sum of the contributions coming from all other particles included in the largest sphere centered on the central particle.

In Fig. 4, we present the numerical results for σ_F versus δ for $\delta < 1/2$ compared with the theoretical prediction for small displacements given by Eqs. (30) and (31). The agreement is excellent up to $\delta \simeq 0.2$. Beyond this value σ_F increases faster than the theoretical prediction for small displacements $\delta \ll 1/2$, and starts to show the diverging behavior for $\delta \to (1/2)^-$ as predicted by Eqs. (58) and (73).

This point is shown better by Fig. 5, where, in order to show the logarithmically diverging behavior of σ_F^2 when $\delta \to (1/2)^-$, we have plotted the numerical results for σ_F^2 versus $(1-2\delta)$, with a logarithmic scale for the latter, for $0.4 < \delta < 0.499$ (and choosing as above $\ell = n_0 = 1$). Indeed, if the approximated theoretical prediction Eq. (73) of a logarithmic divergence is right, this should give a straight line. This prediction is verified by the numerical simulations, albeit with a

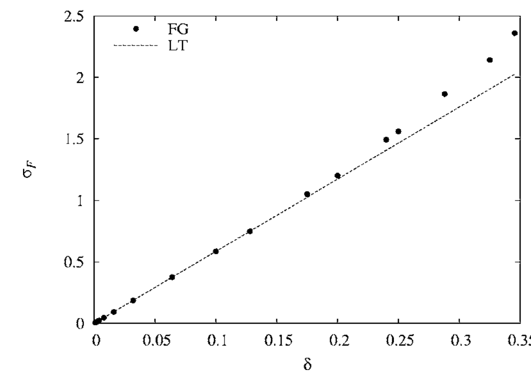

FIG. 4. Behavior of σ_F vs $\delta = \Delta/\ell$ for $p(\mathbf{u})$ given by Eqs. (63) and (64) with $\delta < 1/2$. For $\delta \leq 0.2$, Eqs. (30) and (31) (dashed straight line), which are valid only for $\delta \ll 1/2$, apply very well and the agreement with numerical results (circles) is excellent up to approximately $\delta \simeq 0.2$. For larger values, the actual values of σ_F start to increase faster than this simple linear prediction and the approximation in the manner of Chandrasekhar given in Sec. VII starts to work as shown well by the next figure.

prefactor in the logarithm which is smaller than the theoretical one. This discrepancy can be explained by the strong approximations adopted in Sec. VII to obtain Eq. (73).

In Fig. 6, we report the comparison for the PDF $W(F)$ between the Gaussian theoretical prediction Eq. (59) and σ_F^2 as given by the linear approximation $\langle |\mathbf{F}^{(l)}|^2 \rangle$ of Eq. (30) for an example of the case $\delta \ll 1/2$ (we have chosen $\delta = 0.05$) and the numerical results.

In Fig. 7, on the other hand, we report the numerical evaluation of $W(F)$ for the cases $\delta = 0.4$ and 0.45 versus the Gaussian $W_G(F)$ PDFs with the same variances. As theoretically predicted by Eq. (74), already for $\delta = 0.4$ it starts to be evident that the actual $W(F)$ has a fatter large-F tail and a

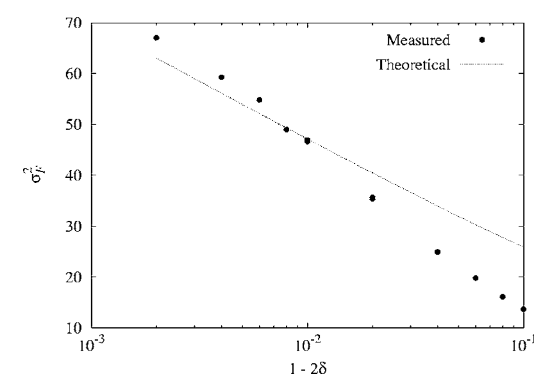

FIG. 5. Numerical results (circles, and best fit given by the dashed line) vs the approximated theoretical prediction (continuous line) Eq. (73) obtained in Sec. VII, with $\ell = n_0 = 1$. The numerical results, as predicted by Eq. (73), show a logarithmic divergence of σ_F^2 in $(1-2\delta)$. However, the slope is about 20% smaller than the approximated theoretical prediction. Considering that this theoretical result is obtained by a strong approximation (which becomes accurate only for larger values of δ), which consists in mimicking the SL with an inhomogeneous Poisson particle distribution with radial density equal to the average conditional density in the SL, we consider this to be a good result.

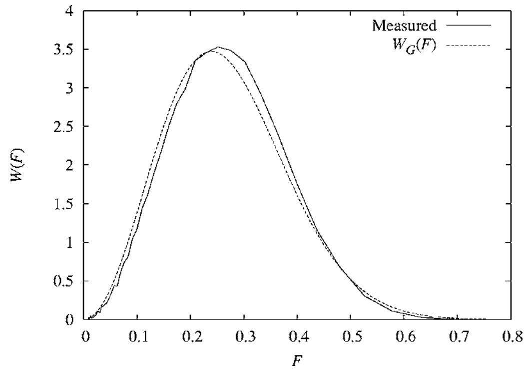

FIG. 6. $W(F)$ found numerically for a SL with $\delta=0.05 \ll 1/2$ and the theoretically predicted Gaussian distribution $W_G(F) = 4\pi F^2 P_G(\mathbf{F})$, where $P_G(\mathbf{F})$ is defined by Eq. (59) with σ_F^2 given by $\langle|\mathbf{F}^{(l)}|^2\rangle$ of Eq. (30). The agreement between the two curves is very good.

peak lower than that of $W_G(F)$. In fact, for such values of δ the kurtosis excess λ_{NG} acquires a significantly positive value. This discrepancy becomes even more clear for $\delta = 0.45$ for which $\lambda_{NG} \approx 2.1$. In fact, in this case the large-F tail starts to develop a power-law feature even though an exponential cutoff is still evident.

The "critical" case $\delta=1/2$ [i.e., where for the first time $W(F)$ develops a power-law large-F tail] is represented in Fig. 8, where the numerical $W(F)$ is compared with the theoretical prediction for the large- and small-F tails, respectively, given by Eqs. (71) and (78). Despite the roughness of the approximation, notably for the small-F amplitude A, the agreement is very good.

The Holtzmark-like case $\delta > 1/2$ for $W(F)$ is represented in Fig. 9 for the particular value $\delta=1$ of the shuffling parameter. It is compared both with the exact Holtzmark distribution obtained in a Poisson particle distribution with the same average number density, and with the theoretical predictions

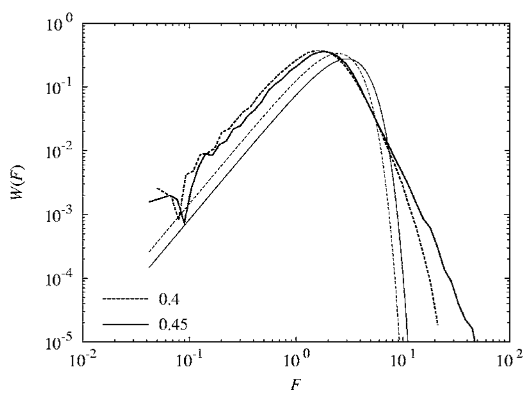

FIG. 7. Comparison between $W(F)$ found numerically for a SL, respectively, with $\delta=0.4$ and 0.45 and the Gaussian distributions $W_G(F) = 4\pi F^2 P_G(\mathbf{F})$ with the same variances $\langle F^2\rangle$. One observes that in both cases, $W(F)$ has its peak at smaller F and a more significant large-F tail than the Gaussian approximation as predicted by Eq. (74) giving a well defined positive kurtosis excess λ_{NG}. The closer δ approaches the "critical" value 1/2, the larger is this deviation.

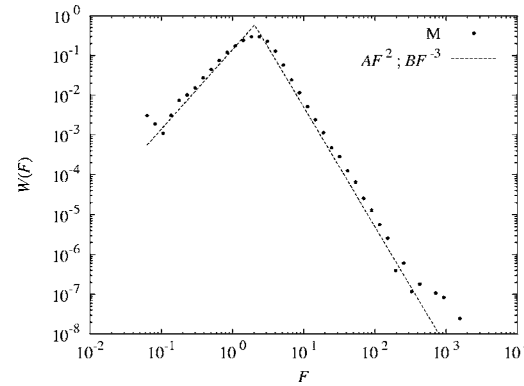

FIG. 8. Numerical $W(F)$ from simulations of computer realizations of the SL with $\delta=1/2$. The large and small F power-law approximations are given, respectively, by Eqs. (71) and (78).

for the large- and small-F tails given, respectively, by Eq. (75), with $\xi(0)$ given by Eq. (76), and Eq. (78). On the one hand, we see that the $W(F)$ approximates quite well the exact Holtzmark one, confirming that the shape of $W(F)$ is mainly determined by the small separation properties of the particle distribution. On the other hand, we see also that our theoretical approximation shows a good agreement with simulations, although the small-F prediction is rougher than the large-F one. This is due to the very simple method we have adopted in Sec. VIII D to evaluate the amplitude of this tail instead of calculating the more precise but difficult Eq. (62).

Finally, as a further test of our theoretical predictions, we have plotted in Fig. 10 the ratio $W(F)/W_H(F)$ giving a measure of the dependence on δ of the large-F tail of $W(F)$ for a wide range of values $\delta > 1/2$. We have compared these values with the theoretical prediction given by Eq. (75), with $\xi(0)$ given by Eq. (76) as functions of δ. The agreement between numerical simulations and theory for this quantity is impressive, particularly so given the nonmonotonous behavior of $\xi(0)$.

IX. DISCUSSION AND CONCLUSIONS

In this paper, we have presented a detailed study of the statistical distribution of the total gravitational (or Coulomb)

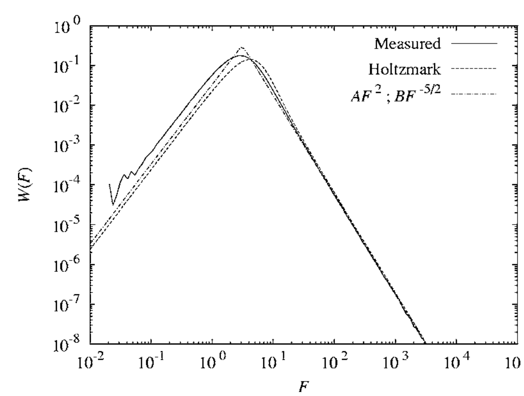

FIG. 9. Comparison between numerical $W(F)$ for the case $\delta = 1$ with the exact Holtzmark distribution and the theoretical predictions given by Eq. (75), with $\xi(0)$ given by Eq. (76), and Eq. (78).

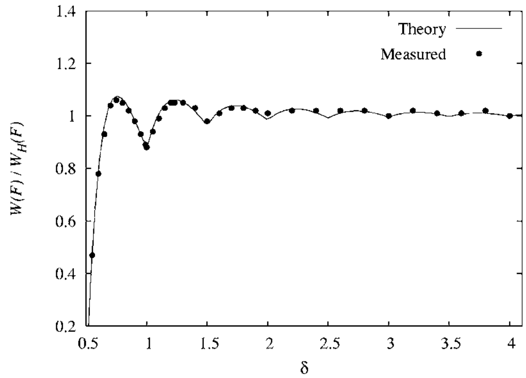

FIG. 10. Plot of $W(F)/W_H(F)$ for a wide range of values $\delta > 1/2$ compared with the theoretical prediction given by Eq. (75), with $\xi(0)$ given by Eq. (76) as functions of δ.

force acting on a particle belonging to a randomly perturbed lattice and due to the sum of the pair gravitational interactions with all the other particles.

In the first part of the paper, we studied the case in which the displacements applied to the lattice particles to produce the perturbed lattice are small. In particular, we analyzed the linear expansion of the force in the displacements. We observed seen that if only displacements strictly smaller than half the lattice cell size are permitted, this linear expansion can be used to calculate to a good approximation the force variance. Otherwise, this variance goes to infinity, due to the small-scale divergence of the pair interaction, even though the average quadratic displacement is kept small. We have seen that in the case in which the force variance is finite, it can be seen as the sum of two different terms: the former comes from the small random displacements from the lattice position of the sources keeping the particle on which the force is calculated fixed at its initial lattice position, and the latter comes from the displacement of this particle from the lattice position keeping at the same time the sources at the initial lattice positions. This second term can also be seen as the contribution of the uniform negative background if it is obtained by summing the contribution with respect to the original lattice position of the particle feeling the force.

In the subsequent sections, we focused our attention on an approximate extension to our case of the Chandrasekhar approach leading to the Holtzmark PDF for the homogeneous Poisson particle distribution. In this way, we have been able to find approximate expressions both for the force PDF and its characteristic function (and the cumulant generating function) for all the range of typical displacements. We have seen that from a qualitative point of view, this functional prediction holds for the whole range of random displacements, and that the agreement becomes quantitatively good for typical displacements of the order of or larger than half the lattice cell size (i.e., when density-density correlations start to be small and the contribution of the Bragg peaks to the particle PS is strongly reduced).

All the above results have then been positively confirmed by a direct comparison with numerical simulation of the system in which the SL particle distributions are generated with a Monte-Carlo-like method and the force probability distribution is numerically computed.

We have underlined that, in general, when δ starts to be of the order of half the lattice cell size, i.e., when the minimal permitted distance between particles vanishes, the force PDF is dominated by the first NN contribution becoming very similar to the Holtzmark distribution $W_H(F)$ even though at large distances the SL particle distribution is still very different from a homogeneous Poisson particle distribution. As has been noted, this is due to the small-scale divergence of the gravitational (or Coulomb) pair interaction between particles. This suggests that, when the minimal permitted distance between particles vanishes, the same behavior for $W(F)$, both at small and large F, is expected to be found in all the spatial particle distributions sharing the same small-scale correlation properties independently of the large-scale features. This can be seen clearly in Eq. (79), where it is shown that while the small and large F exponents of $W(F)$ are universal, the amplitudes depend only on $\xi(0)$.

To conclude, let us finally return briefly to comment on the applications of the results and methods we have just found. They can be useful in various different contexts mentioned in the Introduction, but we will discuss here only the primary application that has motivated our own study, namely the comprehension of the dynamics of self-gravitating systems studied in cosmology. In this context, large numerical "*N*-body" simulations of purely self-gravitating, essentially pointlike[11] particles are used to model the evolution of a self-gravitating fluid. The probability distribution of the force on a given particle is a useful quantity to understand notably in considering (i) the early time dynamics and, more specifically, (ii) questions concerning the effects of discreteness in these simulations (see [16,31]). In [17], we report a full analysis of the dynamics of the gravitational evolution of *N*-body simulations from precisely the SL initial conditions analyzed here, and the results given here will be directly applied in understanding these questions. Much can be understood from this study about the case of real cosmological *N*-body simulations, in which the initial conditions are lattices subject to small *correlated* perturbations. In this respect, we note that the methods developed here, notably the approximate generalization of the Chandrasekhar method, can in principle be generalized to such distributions. The present study of the SL is just a first simpler starting point. Further, the methods used here can be seen as a first example for calculations of other statistical quantities in such distributions of relevance in understanding the dynamics of these self-gravitating systems at larger scales and longer times, e.g., the probability distribution of the force on the center of mass of coarse-grained cells, the two-point correlation functions of gravitational force as a function of separation, etc.

ACKNOWLEDGMENTS

We thank the "Centro Ricerche e Studi E. Fermi" (Rome,

[11]The smoothing generically introduced in the gravitational interaction is relevant only at distances much smaller than the initial interparticle separation.

Italy) for the use of a supercomputer for numerical calculations, the EC Grant No. 517588 "Statistical Physics for Cosmic Structures," and the MIUR-PRIN05 project on "Dynamics and thermodynamics of systems with long range interactions" for financial support. M.J. thanks the Instituto dei Sistemi Complessi for its kind hospitality during October 2006. Finally, we thank L. Pietronero for useful discussions and suggestions.

APPENDIX A: SMALL-q ANALYSIS OF THE CUMULANT GENERATING FUNCTION $\mathcal{G}(\mathbf{q})$

In this appendix, we provide some details of the small-q expansion of the cumulant generating function $\mathcal{G}(\mathbf{q})$ as defined in Eqs. (52) and (53) for $\Delta \geqslant \ell/2$. As explained in Sec. VII, for such values of Δ and sufficiently small x, the correlation function $\xi(x)$ can be written as

$$\xi(x) = \xi(0) + Bx^{\beta} + o(x^{\beta}), \tag{A1}$$

where $\xi(0)=-1$ for $\Delta=\ell/2$ and $-1<\xi(0)<+\infty$ for $\Delta>\ell/2$, and $B,\beta>0$. Let us suppose that the expansion (A1) is valid for $x<x_0$ with $x_0>0$, and rewrite the integral in Eq. (53) as the following sum of two integrals:

$$\mathcal{G}(\mathbf{q}) = -4\pi n_0\left\{\int_0^{x_0} dx x^2[1+\xi(x)]\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right] + \int_{x_0}^{+\infty} dx x^2[1+\xi(x)]\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right]\right\}. \tag{A2}$$

Now in the first integral we can use Eq. (A1), while in the second one, assuming $q \ll x_0^2$, we can expand $\sin(q/x^2)$ in Taylor series. Since $x_0>0$ and independent of q, and $\xi(x)$ vanishes for $x\to+\infty$, it is now simple to show that the second integral is of order q^2 at small q. Let us call $I(q)$ the first integral, i.e.,

$$I(q) = \int_0^{x_0} dx x^2[1+\xi(x)]\left[1-\frac{x^2}{q}\sin\left(\frac{q}{x^2}\right)\right].$$

By using Eq. (A1), we have

$$I(q) = [1+\xi(0)]q^{3/2}\int_0^{x_0/\sqrt{q}} dt t^2[1-t^2\sin(t^{-2})] + Bq^{(3+\beta)/2}\int_0^{x_0/\sqrt{q}} dt t^{2+\beta}[1-t^2\sin(t^{-2})] + o[q^{(3+\beta)/2}], \tag{A3}$$

where we have changed the variable to $t=x/\sqrt{q}$ in both integrals. In Eq. (A3), the first integral converges in the limit $q\to 0$ while the second one converges only for $0<\beta<1$, diverges logarithmically for $\beta=1$, and diverges as $q^{(1-\beta)/2}$ for $\beta>1$. Therefore, we can conclude that for $\xi(0)>-1$, i.e., $\Delta>\ell/2$, in the limit $q\to 0$ up to the dominant term we have

$$\begin{aligned}\mathcal{G}(\mathbf{q}) &\simeq -4\pi n_0 I(q)\\ &\simeq -\left\{4\pi n_0[1+\xi(0)]\int_0^{\infty} dt t^2[1-t^2\sin(t^{-2})]\right\}q^{3/2}\\ &= -n_0[1+\xi(0)]C_H(q).\end{aligned} \tag{A4}$$

Instead for $\xi(0)=-1$, i.e., $\Delta=\ell/2$, the coefficient of the first integral of $I(q)$ vanishes, and the second one [considering also the second integral of Eq. (A2) for $\beta>1$, which is also of order q^2] gives

$$\mathcal{G}(\mathbf{q}) \propto \begin{cases} -q^{(3+\beta)/2} & \text{for } 0<\beta<1\\ q^2\ln q & \text{for } \beta=1\\ -q^2 & \text{for } \beta>1.\end{cases} \tag{A5}$$

Note that even for $\beta>1$, differently from the case $\Delta<\ell/2$, the small-q expansion of $\mathcal{G}(\mathbf{q})$ contains a singular part even though it is of order higher than q^2.

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